Dynamic Contests with Resource Constraints

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Abstract

We study a dynamic contest between two players who compete against each other in n different stages. The players' have common winning values that may vary over the stages as well as heterogeneous resource budgets that decrease within the stages proportionally to the resources allocated in the previous stages. We find a subgame-perfect equilibrium of this dynamic contest and show that when the winning value is equal between the stages, the players' resource allocations are weakly decreasing over the stages. We also study the effect of several distributions of winning values on the players' resource allocations. We show both the distribution of winning values that balances the players' resource allocations and the distribution of winning values that maximizes the players' total resource allocations.

1 Introduction

We consider a dynamic contest in which two firms that produce a homogeneous good are engaged in a competition in selling their products in the same market. The competition is repeated in several stages and the goal of each firm is to maximize the sum of its payoffs over all the stages. The firms' winning values may vary over the different stages. Each of the firms has a resource budget of which part of the resource allocation in the previous stage is completely diminished while part is recycled. We model this dynamic competition to study n-stage contests with two asymmetric players where in each of the stages $1 \leq t \leq n$ the players compete in the Tullock contest (see Tullock 1980) for a prize (value of winning) equal to p_t. The

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players have heterogeneous resource budgets that decrease within the stages proportionally to the resources allocated in the previous stages, such that for each resource unit that a player allocates, he loses $0 \leq \alpha \leq 1$ units of resource from his budget.

The model most related to our dynamic contest is the Colonel Blotto game introduced by Borel (1921). In this game, two players compete against each other in n different contests. Each player distributes a fixed amount of resource over the n contests without knowing his opponent's distribution of the resource. In each contest, the player who allocates the higher level of the resource wins, and each player's payoff is a function of the sum of the wins across the individual contests (see, for example, Snyder 1989, Roberson 2006, Kvasov 2007 and Hart 2008). In particular, the Colonel Blotto game where the players compete in a Tullock contest (see, Friedman 1958, Robson 2005 and Klumpp and Polborn 2006) is similar to our model in the extreme case when the resource budget is completely reduced over the stages ($\alpha = 1$). However, when $0 < \alpha < 1$ there are some differences between our dynamic contest and the static Colonel Blotto game. The most prominent one is that in the Colonel Blotto game the order (timing) of the contests with respect to the players' wining values is not especially important, while in our dynamic model this feature plays a critical role.

One of the more interesting questions in multi-stage contests is how to allocate the players' resources over the different stages of the contest. The contest theory literature offers different opinions about whether or not players strategically allocate their resources in such contests. Using data from professional sport leagues in the US, Ferrall and Smith (1999) showed that teams do not strategically allocate their effort but actually exert as much effort as possible in each of the stages. On the other hand, according to Amegashie et al. (2007) as well as Matros (2006) if players have fixed equal resources they spend more resources in the initial rounds than in the following ones. Likewise, Harbaugh and Klump (2005) showed in a two-stage tournament that weak players exert more effort in the first stage whereas strong players save more effort for the second stage. In our n-stage contest, when the prize for winning is equal between the stages, a player allocates a resource that is weakly decreasing over the stages, while if the value of α is sufficiently high (high fatigue) a player allocates the same level of resource in the first stages and from some stage onwards decreases his resource allocation over the stages. Moreover, for $\alpha \geq 0.5$, independent of the number of stages n, each player allocates the same level of resource over the first $n-1$ stages, but allocates a smaller level of resource in the last stage. We also find that the number of first stages in which the players allocate the same level of resources increases in α (fatigue). The intuition for this is clear since when fatigue is lower, a player has less incentive to save resources such as effort for the next stages and will allocate the highest possible resource according to his resource budget in every stage of the contest.[1]

A prominent application of our dynamic model is a political contest (elections) between two parties who compete in several regions where each region has a different number of delegates. The goal of each party is to maximize its number of delegates. In that case, because of the asymmetry of the regions, the parties' values of winning vary over the different regions. Moreover, if the elections are sequential, the timing of the contest in each region is critical for the parties' resource allocations. Thus, a designer who has control of the order of the elections in the different regions can significantly influence the results of such a political contest. To address this issue, in our multi-stage contest we study the effect of several distributions of winning values over the stages on the players' resource allocations. Moreover, in some multi-stage contests, especially in sport such as round-robin tournaments, the designer's goal may be to balance the effort allocations such that a team will exert the same effort over all the stages. We deal with this issue as well and show that the resource allocation can be balanced such that every player allocates the same resource over all the stages if the players have the same winning value over all the first $n-1$ stages and the winning value in the final stage is sufficiently larger. The intuition for this result is that by awarding a large prize in the last stage, the players have an incentive to save efforts in the previous stages since winning in the last stage is very profitable. An interesting point is that the allocation of prizes for winning that yields balanced resource allocations in our model is different from the one presented in Rosen (1986) according to which the rewards in later stages must be higher than the rewards in earlier stages in order to sustain a non-decreasing effort along the elimination tournament.[2]

In contrast to balancing the resource allocations, in many contests, both one-stage and multi-stage

[1] Ryvkin (2009) also studied the phenomenon of fatigue but in a different multi-stage contest known as the best-of k contest in which he models fatigue as a reduction in a player's probability of winning resulting from previous efforts. He found that agents are more likely to allocate higher resources in the later stages of competition which is exactly opposite to our findings.

[2] Additional works on allocation of resources in sequential contests include, among others, Warneryd (1998), Konrad (2004) and Kovenock and Roberson (2009).

contests, the goal might be to maximize the players' total resource allocations. The optimal allocation of prizes for this purpose has been well studied in the literature. Fu and Lu (2009), for example, studied a multi-stage sequential elimination Tullock contest, and showed that in the optimal contest, a designer who wishes to maximize the players' total resource allocation should eliminate one contestant at each stage until the final, and then the winner of the final takes the entire prize sum. Moldovanu and Sela's (2006) study of an elimination two-stage all-pay auction under incomplete information revealed that it is optimal for a designer who wishes to maximize the expected total effort to allocate the entire prize sum to the winner in the second (final) stage of the tournament. In our dynamic model, however, we find that the maximal total resource allocations over all the stages is obtained when the prizes for winning decrease over the first $n-1$ stages, with the prize in the last stage being either smaller or larger than the previous prizes. This result can be explained by the fact that the highest total effort is obtained when each player allocates a resource in each stage that is equal to his resource budget in that stage such that he does not save resources for the next stages. Since the rate of saving resources is decreasing over the stages, the optimal values of prizes to prevent saving of resources are decreasing over the stages as well.

The rest of the paper is organized as follows: Section 2 introduces our dynamic model. Section 3 analyzes the resource allocations where the players' winning value is the same over all the stages. Section 4 studies the distribution of winning values that yields balanced resource allocations over all the stages, and Section 5 studies the distribution of winning values that yields the highest total resource allocations. Section 6 concludes.

2 The model

Consider a dynamic contest between two players, 1 and 2. The players' value of winning the contest in stage $t, 1 \leq t \leq n$ is p_t. Let $i, j \in \{1, 2\}$. Then, if the players compete against each other in stage t, player i's expected utility in stage t is $u_t^i = p_t \frac{x_t^i}{x_t^i + x_t^j}$ where x_t^i, x_t^j are the players' resource allocations in stage t. Player i has a budget of v^i units of resource to allocate across the n stages. The resource budgets are reduced in the stages such that for each resource unit that a player allocates he loses α units of resource from his budget; that is, $v_{t+1}^i = v_t^i - \alpha x_t^i$, $0 \leq \alpha \leq 1$ where v_t^i is player i's resource budget in stage t. A player's resource allocation in each stage is smaller or equal to his resource budget in that stage. Furthermore, each unit of resource up to the budget constraint has a zero opportunity cost, so that the resource budget is use-it or lose-it. The goal of each player is to maximize his expected total payoff over the n stages. Henceforth, we refer to this dynamic model as an n-stage contest.

3 Decreasing resource allocations

We assume first that the value of winning a match is the same over all the stages and that this value is normalized to be 1. It is important to note that in our n-stage contest, only the relations between the values in the different stages affect the players' resource allocations. In other words, if we multiply the winning values by the same constant, the players' resource allocations will not be changed. Let $i, j \in \{1, 2\}$. Then if the winning value is equal to 1 in all the stages and player i has a resource budget of v^i_m in stage $m, m = 1, ..., n$, his maximization problem in that stage is

$$\begin{aligned} & Max_{x^i_m,...,x^i_n} \sum_{t=m}^{n} \frac{x^i_t}{x^i_t + x^j_t} \qquad (1) \\ & s.t. \\ x^i_t \leq \; & v^i_m - \alpha \sum_{s=m}^{t-1} x^i_s \quad , \quad t = m, ..., n \end{aligned}$$

Since the resource budget is use-it or lose-it, each player in the last stage (stage n) allocates a resource that is equal to his resource budget. The simultaneous solution of both players' maximization problems yields a subgame-perfect equilibrium of the n-stage contest with identical winning values over the n stages.

Proposition 1 *In a subgame-perfect equilibrium of the n-stage contest with identical winning values over the stages, if player $i, i = 1, 2$ has a resource budget of v^i_m in stage $m, m = 1, ..., n$, then his resource allocation in the next stages are as follows:*

1) For $\alpha < \frac{1}{\widetilde{n}}$, $\widetilde{n} = n - m + 1$

$$x^i_t = v^i_m (1 - \alpha)^{t-m} \quad , \quad t = m, ..., n \ , \ i = 1, 2 \qquad (2)$$

such that in every stage $t \geq m$, each player allocates a resource that is equal to his resource budget in that stage.

2) For $\frac{1}{\widetilde{n}+k+2} \leq \alpha < \frac{1}{\widetilde{n}+k+1}, k = 2, 3, ..., \widetilde{n}$

$$\begin{aligned} x_t^i &= \frac{v_m}{\widetilde{n}\alpha} \quad , \ t = m, ..., m+k-1 \ , \ i = 1, 2 \\ x_t^i &= \frac{v(\widetilde{n}+1-k)(1-\alpha)^{t-m-k}}{\widetilde{n}} \quad , \ t = m+k, ..., n \ , \ i = 1, 2 \end{aligned} \tag{3}$$

such that each player's resource allocation is the same over the first $k-1$ *stages and is decreasing over the last* $\widetilde{n}-k+1$ *stages.*

Proof. See Appendix. ∎

The following example illustrates the subgame-perfect equilibrium strategies given by Proposition 1 for the case of the three-stage contest.

Example 1 *In a subgane-perfect equilibrium of the three-stage contest the players' resource allocations for different values of* α *are given by*

$i = 1, 2$	x_1^i	x_2^i	x_3^i
$\alpha < \frac{1}{3}$	v^i	$v^i(1-\alpha)$	$v^i(1-\alpha)^2$
$\frac{1}{3} < \alpha \leq \frac{1}{2}$	$\frac{v^i}{3\alpha}$	$\frac{2v^i}{3}$	$\frac{2v^i(1-\alpha)}{3}$
$\alpha > \frac{1}{2}$	$\frac{v^i}{3\alpha}$	$\frac{v^i}{3\alpha}$	$\frac{v^i}{3}$

Figure 1 shows that for every value of α, $x_1^i(\alpha) \geq x_2^i(\alpha) \geq x_3^i(\alpha)$, $i = 1, 2$, *i.e., player* i*'s resource allocation decreases during the stages. We can also see that player* i*'s resource allocation in every stage* $x_t^i(\alpha), t = 1, 2, 3$ *is non-increasing in* α.

By Proposition 1, as well as in the above example, the equilibrium strategy forms a non-increasing sequence of resource allocations $x_t^i \geq x_{t+1}^i$, $t = 1, ..., n$, $i = 1, 2$ over all the stages. In addition, for every level of α there is a critical stage $t^*(\alpha)$ such that in all the stages $t \geq t^*(\alpha)$ each player allocates a resource equal to his resource budget in that stage, while in all the previous stages $t < t^*$, each player allocates the same resource which is smaller than his resource budget in that stage.[3]

[3] Similar results were obtained by Che and Gale (1997) for one-stage Tullock contests where each player has a different budget constraint.

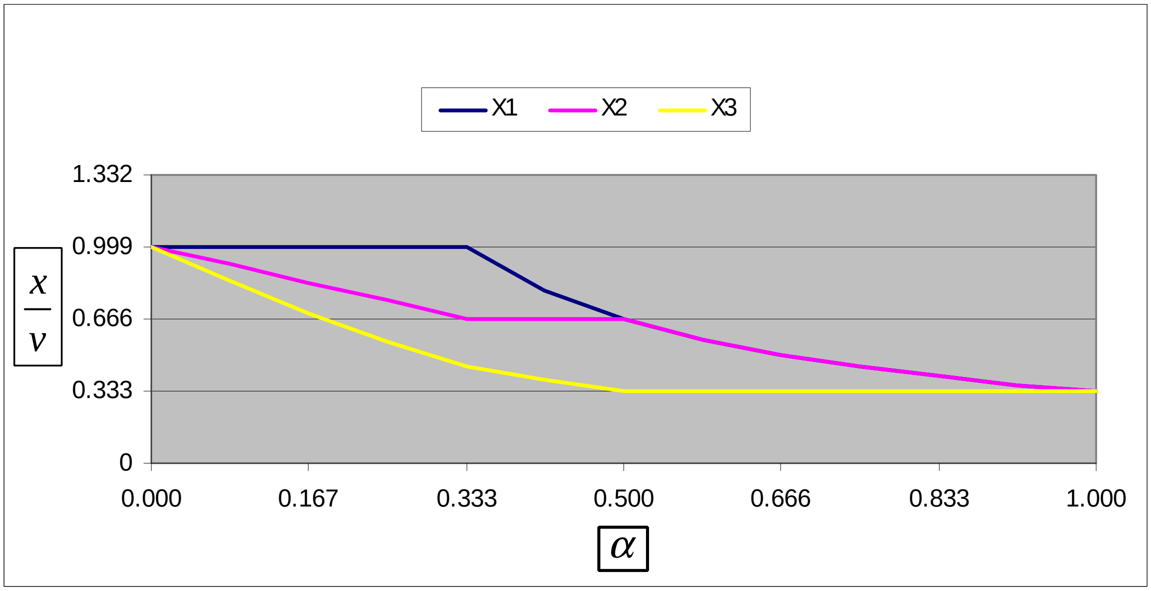

Figure 1: The resource allocation in the three-stage contest

Player i's total resource allocation in the n-atage contest is given for every $\frac{1}{n-k+2} \leq \alpha < \frac{1}{n-k+1}, k = 2, 3, ..., n$ by

$$E_n^i = v^i \frac{k-1}{n\alpha} + \sum_{t=k}^{n} v^i \frac{(n+1-k)(1-\alpha)^{t-k}}{n} = v^i \frac{n-(n+1-k)(1-\alpha)^{n+1-k}}{n\alpha}, \ i = 1, 2 \tag{4}$$

In the $(n+1)$-stage contest, player i's total resource allocation for the same α (note that k is now larger by 1) is given by

$$E_{n+1}^i = v^i \frac{n+1-(n+1-k)(1-\alpha)^{n+1-k}}{(n+1)\alpha}, \ i = 1, 2$$

The difference in player i's total resource allocation between the $(n+1)$-stage and the n-stage contests is

$$E_{n+1}^i - E_n^i = v^i \frac{(n+1-k)(1-\alpha)^{n+1-k}}{n(n+1)\alpha} > 0 \ , \ i = 1, 2$$

Hence, player i's total resource allocation, $i = 1, 2$ increases in the number of stages.

4 Balanced resource allocations

In the previous section, we showed that each player allocates the same level of resource in the first stages of the n-stage contest but decreases his resource allocation in the last stages. It is of interest to examine the distribution of winning values over the stages that balance the players' resource allocations in the contest where each player's resource allocation is the same over all the stages. In a different form of an elimination tournament, Rosen (1986) showed that a series of increasing rewards sustain a non-decreasing resource (effort) allocation along the tournament. In contrast, we show below that in our n-stage contest, the players' resource allocations over all the stages are balanced if the winning value is the same for all the stages except the last one.

If player $i, i = 1, 2$ has a resource budget of v_m^i in stage m, $m = 1, ..., n$ and his resource allocation is the same over all the last $\widetilde{n} = n - m + 1$ stages, we have

$$x_m^i = x_n^i = v_m^i - \alpha \sum_{t=m}^{n-1} x_t^i = v_m^i - \alpha(\widetilde{n} - 1)x_m^i \ , \ i = 1, 2$$

Thus, the level of resource in each of the last $\widetilde{n}$ stages is

$$x_m^i = \frac{v_m^i}{1 + \alpha(\widetilde{n} - 1)} \ , \ i = 1, 2 \tag{5}$$

Player i will now allocate the same resource (given in (5)) over all the stages if the winning value in the last stage is sufficiently larger than the winning value in the other stages. Then, winning in the last stage becomes very profitable for player i, and instead of completely using up his resource budget in the first $n - 1$ stages, he will save part of it for the last stage. Formally, assume that the winning value is identical and normalized to 1 for each of the first $n - 1$ stages, and the winning value in the last stage is p_n. Let $i, j \in \{1, 2\}$. Then if player i has a resource budget of v_m^i in stage $m, m = 1, ..., n - 1$ his maximization problem in that stage is

$$Max_{x_m^i, ..., x_n^i} \sum_{t=m}^{n-1} \frac{x_t^i}{x_t^i + x_t^j} + p_n \frac{x_n^i}{x_n^i + x_n^j} \tag{6}$$

$$s.t.$$

$$x_t^i \leq v_m^i - \alpha \sum_{s=m}^{t-1} x_s^i \ , \ t = m, ..., n$$

The simultaneous solution of both players' maximization problems yields a subgame-perfect equilibrium in which the players' resource allocations are balanced over all the stages of the contest.

Proposition 2 *In a subgame-perfect equilibrium of the n-stage contest, if the value of winning each of the first $n - 1$ stages is equal to 1 and the value of winning the last stage is equal to $\frac{1}{\alpha}$, each player allocates the* *same level of resource over all the stages.*

Proof. See Appendix. ■

It is worth examining the relation between balanced resource allocations and maximal resource allocations. The following result demonstrates that the total resource allocations cannot be maximized under the constraint of balanced resource allocations.

Proposition 3 *The total resource allocations in the subgame-perfect equilibrium of the n-stage contest given by Proposition 2 when the players' resource allocations are the same over all the stages, is smaller than in the subgame-perfect equilibrium of the n-stage contest given by Proposition 1 when the winning value is the same over all the stages.*

Proof. See Appendix. ■

Proposition 3 invites the question of which distribution of winning values yields the highest total resource allocations. This issue is the focus of the next section.

5 Maximal resource allocations

Assume that the players' winning values are $p_t, t = 1, ..., n$. Let $i, j \in \{1, 2\}$. Then if player i has a resource budget of v_m^i in stage $m, m = 1, ..., n$ his maximization problem in that stage is

$$\begin{aligned} & Max_{x_m^i,...,x_n^i} \sum_{t=m}^{n} \frac{p_t x_t^i}{x_t^i + x_t^j} \\ & s.t. \\ x_t^i & \leq v_m^i - \alpha \sum_{s=m}^{t-1} x_s^i \quad , \quad t = m, ..., n \end{aligned}$$

The following result provides the minimal levels of winning values such that in a subgame-perfect equilibrium the players allocate the highest possible resources in every stage of the contest.

Proposition 4 *In a subgame-perfect equilibrium of the n-stage contest, each player allocates the highest total resource if the winning values satisfy*

$$\begin{aligned} p_n & = 1 \\ p_t & \geq \frac{\alpha}{(1-\alpha)^{n-t}} \ , \ t = 1, ..., n-1 \end{aligned}$$

Proof. See Appendix. ■

By Proposition 4, the sequence of the winning values in the first $n-1$ stages is decreasing such that $p_t = \frac{p_{t+1}}{1-\alpha}$,$1 \leq t \leq n-1$. However, the winning value in the last stage might be larger or smaller than the other values depending on the parameter α. For sufficiently small values of α (low fatigue), all the winning values in the $n-1$ first stages will be smaller than the winning value in the last stage, and for sufficiently large values of α (high fatigue) they will be larger than the winning value in the last stage.

Example 2 *In a subgame-perfect equilibrium of the three-stage contest, the highest total resource allocations are obtained if the winning values satisfy*

$$\begin{aligned} p_3 &= 1 \\ p_2 &\geq \frac{\alpha}{1-\alpha} \\ p_1 &\geq \frac{\alpha(1+p_2)}{1-\alpha} \end{aligned}$$

It can be easily verified that $p_1 \geq p_2$. Moreover, by Figure 2 we can see that the difference between the values $p_1 - p_2$ increases in α, and both values p_1 and p_2 are smaller than p_3 for small values of α but are larger than p_3 for high values.

The last example suggests that there is a conflict if, on the one hand, we want to balance of the players' resource allocations, but on the other, we want to maximize their total resource allocations. To elaborate, by Proposition 4, in order to maximize the total resource allocations, the minimal winning value in the first $n-1$ stages should be $\frac{\alpha}{(1-\alpha)^{n-1}}$, where the value in the last stage is equal to 1. Then we obtain that the the ratio between the last stage's value of winning and the other winning values is $\frac{1}{\frac{\alpha}{(1-\alpha)^{n-1}}} = \frac{(1-\alpha)^{n-1}}{\alpha} < \frac{1}{\alpha}$, which is insufficient for balancing the players' resource allocations over all the stages of the contest.

6 Concluding remarks

We studied a dynamic contest with two asymmetric players who compete against each other in n different stages. Each player has a resource budget that decreases within the stages proportionally to the resource allocation in the previous stages. We showed that if the players' winning values are identical over the stages,

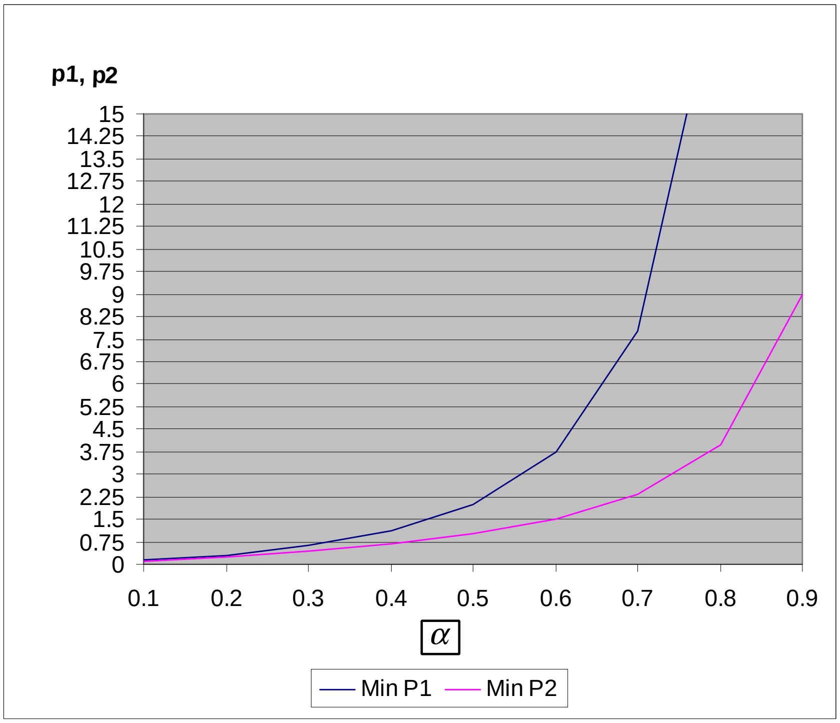

Figure 2: The optimal values of winning in the three-stage contest

then their resource allocations are necessarily decreasing from some stage onwards. On the other hand, if the winning values in all the stages except the last one are identical and the value of winning in the last stage is sufficiently larger than the other winning values, then the players' resource allocations will be balanced such that each player's resource allocation will be the same over all the stages. We also showed that the players' resource allocations will be maximized only if the players' winning values are decreasing over the first $n-1$ stages. These results were obtained for asymmetric players who have different resource budgets but the same winning values in all the n stages of the contest. The generalization of these results to asymmetric n-stage contests where players have different winning values is an interesting objective for future research.

7 Appendix

7.1 Proof of Proposition 1

Let $i, j \in \{1,2\}$. Then, if player i has a resource budget of v^i_m in stage $m, m = 1, ..., n$ his maximization problem in that stage is

$$
\begin{aligned}
& Max_{x^i_m,...,x^i_n} \sum_{t=m}^{n} \frac{x^i_t}{x^i_t + x^j_t} \qquad (7)\\
& s.t. \\
x^i_t \quad \leq \quad & v^i_m - \alpha \sum_{s=m}^{t-1} x^i_s \quad , \quad t = m, ..., n
\end{aligned}
$$

The simultaneous solution of the above maximization problems for both players yields the resource allocations in a subgame-perfect equilibrium of the n-stage contest. Let $\widetilde{n} = n - m + 1$ and $\frac{1}{\widetilde{n}-k+2} \leq \alpha < \frac{1}{\widetilde{n}-k+1}, k = 2, 3, ..., \widetilde{n}$. Assume that the last $\widetilde{n} - k + 1$ constraints are binding while the first $k - 1$ constraints are not (this assumption will be confirmed in the following). Then, player i's Lagrangian is given by

$$
L^i = \sum_{t=m}^{n} \frac{x^i_t}{x^i_t + x^j_t} - \sum_{t=m+k}^{n} \lambda^i_t (x^i_t - v^i_m + \alpha \sum_{s=m}^{t-1} x^i_s) \ , \ i = 1, 2
$$

where $\lambda^i_t, t = m + k, ..., n$ are player i's Lagrangian multipliers. The first-order conditions are

$$
\begin{aligned}
\frac{dL^i}{dx^i_t} &= \frac{x^j_t}{(x^i_t + x^j_t)^2} - \alpha \sum_{s=m+k}^{n} \lambda^i_s = 0 \ , \ t = m, ..., m + k - 1 \ , \ i = 1, 2 \qquad (8)\\
\frac{dL^i}{dx^i_t} &= \frac{x^j_t}{(x^i_t + x^j_t)^2} - \lambda^i_t - \alpha \sum_{s=t+1}^{n} \lambda^i_t = 0 \ , \ t = m + k, ..., n \ , \ i = 1, 2 \\
\frac{dL^i}{d\lambda_t} &= x^i_t - v^i_m + \alpha \sum_{s=m}^{t-1} x^i_t = 0 \ , \ t = m + k, ..., n \ , \ i = 1, 2
\end{aligned}
$$

If the first $k - 1$ constraints are not binding in the maximization problems of both players, then player i's resource allocation in each of the first $k - 1$ stages is given by

$$
x^i_t = \frac{v^i_m}{\widetilde{n}\alpha} \quad , \quad t = m, ..., m + k - 1 \ , \ i = 1, 2 \qquad (9)
$$

Then by (9), the first constraint in the maximization problem (7) $\frac{v^i_m}{\widetilde{n}\alpha} < v^i_m$ is satisfied iff $\alpha > \frac{1}{\widetilde{n}}$ and the second constraint $\frac{v^i_m}{\widetilde{n}\alpha} < v^i_m - \alpha \frac{v^i_m}{\widetilde{n}\alpha} = \frac{v^i_m(\widetilde{n}-1)\alpha}{\widetilde{n}\alpha}$ is satisfied iff $\alpha > \frac{1}{\widetilde{n}-1}$. Similarly, the constraint $t, 3 \leq t \leq k-1$, $\frac{v^i_m}{\widetilde{n}\alpha} < v^i_m - (t-1)\alpha \frac{v^i_m}{\widetilde{n}\alpha}$ is satisfied iff $\alpha > \frac{1}{\widetilde{n}+1-t}$. Thus, our assumption that $\frac{1}{\widetilde{n}-k+2} \leq \alpha < \frac{1}{\widetilde{n}+1-k}$ is a necessary and sufficient condition that exactly the first $k - 1$ constraints are not binding.

If the last $\widetilde{n}-k+1$ constraints are binding, then by (9) we obtain that player i's resource allocation in the last $\widetilde{n}-k+1$ stages is given by

$$x_t^i = v_m^i - \alpha \sum_{s=m}^{t-1} x_s^i = \frac{(\widetilde{n}+1-k)v_m^i(1-\alpha)^{t-m-k}}{\widetilde{n}} \quad t = m+k, ..., n \ , \ i = 1,2$$

Using the first order conditions (8) we obtain that

$$\lambda_n^i = \frac{v_m^i \widetilde{n}}{(v_m^i + v_m^j)^2(\widetilde{n}+1-k)(1-\alpha)^{n-m-k}} > 0$$

and

$$\lambda_t^i = \frac{v_m^i \widetilde{n}\alpha(1-a)^{n-t-1}(1-(n-t-1)a)}{(v_m^i + v_m^j)^2(\widetilde{n}+1-k)(1-\alpha)^{n-m-k}} \quad t = m+k, ..., n-1$$

Since $\alpha < \frac{1}{\widetilde{n}+1-k} = \frac{1}{n-m+2-k}$ we obtain that $\lambda_t^i > 0$ for all $t = m+k, ..., n$ and therefore the last $\widetilde{n}-k+1$ constraints are binding.

Similarly to the above results, if $\alpha \leq \frac{1}{\widetilde{n}}$ we obtain that all the $\widetilde{n}$ constraints in the maximization problem (7) are binding, and the solution of player i's maximization problem is

$$x_t^i = v_m^i - \alpha \sum_{s=m}^{t-1} x_s^i = v_m^i(1-\alpha)^{t-m} \quad t = m, ..., n \ , \ i = 1,2$$

Q.E.D.

7.2 Proof of Proposition 2

Let $i, j \in \{1,2\}$. Then if player i has a resource budget of v_m^i in stage $m, m = 1, ..., n-1$, his maximization problem in that stage is

$$Max_{x_m^i,...,x_n^i} \sum_{t=m}^{n-1} \frac{x_t^i}{x_t^i + x_t^j} + p_n \frac{x_n^i}{x_n^i + x_n^j} \tag{10}$$

$$s.t.$$

$$x_t^i \leq v_m^i - \alpha \sum_{s=m}^{t-1} x_s^i \ , \ t = m, ..., n$$

Let $\widetilde{n} = n-m+1$. We assume that all the first $\widetilde{n}-1$ constraints are not binding but the last constraint is (this assumption will be confirmed in the following). Then, player i's Lagrangian is given by

$$L^i = \sum_{t=m}^{n-1} \frac{x_t^i}{x_t^i + x_t^j} + p_n \frac{x_n^i}{x_n^i + x_n^j} - \lambda_n^i(x_n^i - v_m^i + \alpha \sum_{s=m}^{n-1} x_s^i) \ , \ i = 1,2$$

where λ_n^i is the Lagrangian multiplier. The first-order conditions are

$$\begin{aligned}\frac{dL^i}{dx_t^i} &= \frac{x_t^j}{(x_t^i + x_t^j)^2} - \lambda_n^i \alpha = 0 \ , \ \ t = m, ..., n-1 \ \ , \ \ i = 1, 2 \\ \frac{dL^i}{dx_n^i} &= p_n \frac{x_n^j}{(x_n^i + x_n^j)^2} - \lambda_n^i = 0 \ , \ \ i = 1, 2\end{aligned} \tag{11}$$

From the comparison of the first $\widetilde{n} - 1$ first-order conditions for both players we have

$$x_m^i = ... = x_{n-1}^i \ , \ i = 1, 2$$

If we divide the first first-order condition by the last one we obtain

$$\frac{x_m^i}{p_n x_n^i} = \alpha \ , \ \ i = 1, 2$$

If $x_n^i = x_m^i$, the value of winning in the last period is equal to

$$p_n = \frac{1}{\alpha}$$

By the last first-order condition given by (11) we obtain that

$$x^i = x_n^i = v_m^i - \alpha \sum_{t=m}^{n-1} x_t^i = v_m^i - \alpha(\widetilde{n} - 1)x^i$$

Thus, the level of resource in each stage is given by

$$x^i = \frac{v_m^i}{1 + \alpha(\widetilde{n} - 1)} \ , \ i = 1, 2$$

In order to show that all the first $\widetilde{n} - 1$ constraints in the maximization problem (10) are not binding, it is sufficient to show that

$$x_{n-1}^i = \frac{v_m^i}{1 + \alpha(\widetilde{n} - 1)} \leq v_m^i - \alpha \sum_{s=m}^{n-2} x_s^i = \frac{v_m^i(\alpha + 1)}{1 + \alpha(\widetilde{n} - 1)} \ , \ i = 1, 2$$

Thus, all the first $\widetilde{n} - 1$ constraints are not binding for $\alpha > 0$ and since

$$\lambda_n^i = \frac{v_m^j(1 + \alpha(\widetilde{n} - 1))}{\alpha(v_m^i + v_m^j)^2} > 0 \ , \ i = 1, 2$$

the last constraint is binding accordingly to our assumption. *Q.E.D.*

7.3 Proof of Proposition 3

By (4), player i's total resource allocation in the contest with equal winning values over the stages (when each value is normalized to 1) is

$$E_p^i(k) = v^i \frac{n-(n+1-k)(1-\alpha)^{n+1-k}}{n\alpha} \quad i = 1, 2$$

where $\frac{1}{n-k+2} \leq \alpha < \frac{1}{n-k+1}$, $k = 2, ..., n$ (for $k = 1$, $0 \leq \alpha < \frac{1}{n}$).

By (5), player i's total resource allocation in a contest with equal levels of resource over the stages is

$$E_e^i = v^i \frac{n}{1+\alpha(n-1)} \quad i = 1, 2$$

For $k = n$ we have

$$E_p^i(k) - E_e^i = v^i \left(\frac{n-(1-\alpha)}{n\alpha} - \frac{n}{1+\alpha(n-1)}\right) = v^i \frac{(n-1)(1-\alpha)^2}{n\alpha(1+\alpha(n-1))} > 0\ ,\ i = 1, 2$$

Suppose that $E_p^i(k) - E_e^i > 0$ for all $n \geq k \geq \widetilde{k}$. We will show by induction that this inequality holds for $k = \widetilde{k} - 1$. The difference between a player i's resource allocation for $k = \widetilde{k} - 1$ is

$$\begin{aligned} E_p^i(\widetilde{k}-1) - E_e^i &= v^i \left(\frac{n-(n-\widetilde{k}+2)(1-\alpha)^{n-\widetilde{k}+2}}{n\alpha} - \frac{n}{1+\alpha(n-1)}\right) \\ &= v^i \left(\frac{n-(n-\widetilde{k}+1)(1-\alpha)^{n-\widetilde{k}+1}}{n\alpha} - \frac{n}{1+\alpha(n-1)}\right) + \frac{(1-\alpha)(\alpha(n-\widetilde{k}+1)+\alpha-1)}{n\alpha} \end{aligned}$$

By the induction assumption, we need to show that

$$\frac{(1-\alpha)(\alpha(n-\widetilde{k}+1)+\alpha-1)}{n\alpha} \geq 0$$

Thus, it is sufficient to show that

$$\alpha(n-\widetilde{k}+2) - 1 \geq 0$$

Since by definition $\frac{1}{n-k+2} \leq \alpha < \frac{1}{n-k+1}$, we obtain that the last inequality holds. *Q.E.D.*

7.4 Proof of Proposition 4

We first show that the maximal total resource allocation of each player is obtained when he allocates the highest possible resource in every stage; namely, each player allocates a resource in every stage that is equal to his resource budget in that stage. The maximization problem of player $i's$ total resource allocation, $i = 1, 2$, is

$$Max_{x_1,...,x_n} \sum_{t=1}^{n} x_t^i$$
$$s.t.$$
$$x_t^i \leq v^i - \alpha \sum_{s=1}^{t-1} x_s^i \ , \ t = 1, ..., n$$

We assume that all the constraints are binding (this assumption will be confirmed in the following). Player i's Lagrangian is given by

$$L_1^i = \sum_{t=1}^{n} x_t^i - \sum_{t=1}^{n} \delta_t^i (x_t^i - v^i + \alpha \sum_{s=1}^{t-1} x_s^i) \ , \ i = 1, 2$$

where $\delta_t^i, t = 1, ..., n$ are the player i's Lagrangian multipliers. The first-order conditions are

$$\frac{dL_1^i}{dx_t} = 1 - \alpha - \delta_t^i - \alpha \sum_{s=t+1}^{n} \delta_s^i = 0 \ , \ t = 1, ..., n \ , \ i = 1, 2$$
$$\frac{dL_1^i}{d\delta_t^i} = x_t^i - v^i + \alpha \sum_{s=1}^{t-1} x_s^i = 0 \ , \ t = 1, ..., n \ , \ i = 1, 2$$

If all the constraints are binding, the solution of these systems of equations is

$$x_t^i = v^i (1 - \alpha)^{t-1} \ , t = 1, ..., n \ , \ i = 1, 2 \tag{12}$$
$$\delta_t^i = (1 - \alpha)^{n+1-t} \ , t = 1, ..., n \ , \ i = 1, 2$$

Since $\delta_t^i > 0$ for all $t = 1, ..., n$ we obtain the desired result. Next, we find the values of winning such that each player will allocate in every stage a resource that is equal to his resource budget in that stage. Let $i, j \in \{1, 2\}$. If player i has a resource budget of v_m^i in stage $m, m = 1, ..., n - 1$, his maximization problem in that stage is

$$Max_{x_m^i,...,x_n^i} \sum_{t=m}^{n} \frac{p_t x_t^i}{x_t^i + x_t^j} \tag{13}$$
$$s.t.$$
$$x_t^i \leq v_m^i - \alpha \sum_{s=m}^{t-1} x_s^i \ , \ t = m, ..., n$$

The simultaneous solution of the above maximization problems for both players yields the equilibrium resource allocation in a subgame-perfect equilibrium. Player i's Lagrangian is given by

$$L_2^i = \sum_{t=m}^{n} \frac{p_t x_t^i}{x_t^i + x_t^j} - \sum_{t=m}^{n} \lambda_t^i (x_t^i - v_m^i + \alpha \sum_{s=m}^{t-1} x_s^i) \ , \ i = 1, 2$$

The first-order conditions are

$$\begin{aligned} \frac{dL_2^i}{dx_t^i} &= \frac{p_t x_t^j}{(x_j^i + x_t^j)^2} - \lambda_t^i - \alpha \sum_{s=t+1}^{n} \lambda_s^i = 0 \quad , \quad t = m, ..., n \ , \ i = 1, 2 \\ \frac{dL_2^i}{d\lambda_t} &= x_t^i - v_m^i + \alpha \sum_{s=m}^{t-1} x_s^i = 0 \quad , \quad t = m, ..., n \ , \ i = 1, 2 \end{aligned} \tag{14}$$

If all the constraints are binding, player i's resource allocation over all the $\widetilde{n} = n - m + 1$ stages is given by

$$x_t^i = v_m^i - \alpha \sum_{s=m}^{t-1} x_s^i = v_m^i (1 - \alpha)^{t-m} \quad , \ t = m, ..., n \ , \ i = 1, 2 \tag{15}$$

By (14) and (15) we obtain that

$$\lambda_n^i = \frac{v_m^j}{(v_m^i + v_m^j)^2 (1 - \alpha)^{n-m}} > 0$$

and by induction we obtain that if for all $m \leq t < n - 2$

$$\begin{aligned} \lambda_t^i &= \frac{p_t v_m^j}{(v_m^i + v_m^j)^2 (1 - \alpha)^{t-m}} - \frac{\alpha v_m^j}{(v_m^i + v_m^j)^2 (1 - \alpha)^{n-m}} = \frac{v_m^j (\ p_t (1 - \alpha)^{n-t} - \alpha)}{(v_m^i + v_m^j)^2 (1 - \alpha)^{n-m}} \ , \ i = 1, 2 \\ p_t &= \frac{\alpha}{(1 - \alpha)^{n-t}} \end{aligned}$$

Then

$$\lambda_{t+1}^i = \frac{v_m^j (\ p_{t+1} (1 - \alpha)^{n-(t+1)} - \alpha)}{(v_m^i + v_m^j)^2 (1 - \alpha)^{n-m}} \ , \ i = 1, 2$$

where

$$\lambda_{t+1}^i \geq 0 \text{ iff } p_{t+1} \geq \frac{\alpha}{(1 - \alpha)^{n-(t+1)}} \ , \ i = 1, 2$$

Hence, if the winning values satisfy

$$\begin{aligned} p_n &= 1 \\ p_t &= \frac{\alpha}{(1 - \alpha)^{n-t}} \ , \ t = m, ..., n \end{aligned}$$

all the constraints in the maximization problem (13) are binding. *Q.E.D.*

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