The All-pay Auction with Handicaps*

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Abstract

This paper derives the equilibrium of an all-pay auction with handicaps. This auction provides new insights into the effects of heterogeneity among participants on their behavior in a tournament. We apply these results to static contests with participants that differ in their ability. We also use these results to derive the equilibrium of a two-stage all-pay auction, which corresponds to a dynamic tournament where the heterogeneity arises in the second period from differences in past performance. We show that effort is highest for participants in intermediate positions. The results are helpful to understand the incentives to engage in preemptive bidding in the early periods of a dynamic tournament.

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1 Introduction

The allocation of economic resources often creates incentives that take the form of a tournament. In sporting competitions, prizes are awarded to the athletes that obtain the highest scores; in organizations, workers compete for a job promotion; firms are often involved in R&D contests, or bid for procurement contracts that involve substantial initial investments in developing design proposals; politicians and political parties compete to be elected. In many of these cases, the result of the competition is determined by the actions of the participants over multiple periods. This paper derives the equilibrium of an all-pay auction with handicaps, and shows how this auction can help us understand the behaior of participants in such dynamic contests.

There is an extensive body of theoretical work on tournaments and relative performance evaluation (RPE) beginning with the pioneering work of Lazear and Rosen (1981) and Holmstrom (1982). The main theme of this literature relates to the role of RPE incentive schemes in eliminating common shocks affecting agents' performance, and hence reducing the cost of providing incentives. However, these studies have confined themselves to settings with homogeneous participants reaching a symmetric equilibrium.[1] Yet heterogeneity is ubiquitous in most tournaments, both because participants may be different ex-ante (perhaps because they differ in ability, and hence, the probability of winning), but also because in multi-period tournaments

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[1] An exception is Meyer (1992), who considers a biased dynamic tournament. In her model, participants are heterogeneous. Nonetheless, her assumptions (namely the presence of two participants, and the symmetry in the distribution of the shocks) guarantees the equilibrium is still symmetric.

heterogeneity develops as time lapses, and differences in past performance arise.

Asymmetric tournaments have been studied by modelling them as an all-pay auction in which participants vary in their valuation of the prizes, or equivalently, on the marginal cost of bidding (See Baye et al., 1996; Clark and Riis, 1998; Moldovanu and Sela, 2001). A bid in these models corresponds to a choice of a level of effort (which all must pay regardless of their final outcome). While this has brought some important insights for contest design, differences in the cost of exerting effort do not correspond well with the heterogeneity that becomes important in a dynamic tournament, as the participants learn their interim performance. When prizes are allocated on the basis of total performance over several periods, any advantage gained by a participant in the past will mean a competitor would have to outperform that advantage in order to win.

In this paper we make some progress towards the understanding of the heterogeneity arising from the dynamic nature of a contest. We model this by considering an all-pay auction with handicaps. Such handicaps relate to a player's past realization of output, and allow us to understand the effects of past performance on behavior.[2] We derive the equilibrium in this auction when there are several identical prizes, and show that the effort levels have an inverted U-shape. Effort is lowest for those players that have done very well or very badly in the past. And it is highest for the marginal player that is just about to enter a winning position.

We also apply these results to a dynamic all-pay auction, in which participants have the option to bid over two periods, with the initial bids being observed before

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[2] This all-pay auction can be thought of as a model for the last period of a multi-stage tournament.

the play of the last round. We derive the equilibria of this auction, which offers new insights on the behavior of participants in dynamic contests.

In particular, we show that when the tournament weights the performance of both periods equally, the participants are indifferent between bidding in any of them. The benefits of increasing the lead in the first period are exactly offset by the cost of the bid, leaving players indifferent between bidding in the first or second period. Hence, there is no strategic reason for preemptive bidding. This leads to a multiplicity of equilibria, in which various degrees of first-period bidding can be sustained. Instead, when the performance measure weights the last period performance by more than the first, players prefer to delay their bidding until the last period, and no bidding takes place in the first. The outcome of the auction is therefore the same as in a single period contest. Finally, when the performance measure weights the first period performance by more than the last, players have an incentive to start bidding early. We show that this may lead to an equilibrium with too little effort, where some players bid early and manage to deter further bidding from other competitors in the second round. As a result, these players manage to obtain a positive rent from the auction.

These results suggest that any preemptive bidding in the first period to deter competitors in the second cannot arise with perfect information and linear costs of effort. First period bids only take place when the contest explicitly rewards the players for doing so. Therefore, in the absence of such incentives, early bidding can only be the result of a desire to smooth costs when these are convex, or to signal high ability when there is imperfect information.

Our paper contributes to the tournament literature by gauging the incentive effects of the heterogeneity induced by the dynamic nature of multiperiod contests. Our results suggest that contestant effort depends on the interim ranking, being highest for those in intermediate positions and lowest for those at the extremes of performance.[3] The results also apply to static tournaments when the ability of players affects their performance linearly, rather than determining their marginal cost of providing effort (as in previous work).

The results also have applications to political economy. Several authors have modelled electoral competitions as all-pay auctions. And there are reasons to believe that heterogeneity is important in these settings as well. Eyster and Kittsteiner (2007), for instance, have a model of electoral competition in which the candidates in each constituency play an all-pay auction with handicaps, where the handicaps are given by the distance between the strategy of the party each candidate belongs to, and the preferences of the median voter in that particular constituency. Our results can be used to extend these settings to allow for multiple prizes (if for instance, parties value not just winning an election, but also the number of elected members), and multiple parties.

Closest to this paper is the work of Sela (2009a, 2009b). Sela (2009a) studies an all-pay auction with three periods and two players, where the winner is the best-of-three (the player who wins twice). The players are heterogeneous, but they differ in ther marginal cost of bidding (or the valuation for the prize). Moreover, although winning the first "match" gives you an advantage. This is different from our handi-

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[3] See Casas-Arce and Martínez-Jerez (2009) for evidence of the relevance of these effects.

capps, as the advantage is fully dissipated by losing the second match, regardless of the losing distance. Sela (2009b) studies an all-pay auction with two prizes, where the prizes are awarded sequentially in two independent auctions. Each player can therefore win nothing, one prize or two prizes. Instead, in our auction, there are two prizes, but each player can only win one. Furthermore, he also models heterogeneity assuming differences in the marginal cost of bidding (or the valuation for the prize).

Our set-up also differs from the papers that study elimination tournaments (most notably, Rosen, 1986). In these models, effort in a given period only affects the probability of winning in the current period, whereas in a dynamic tournament, effort in any given period affects the probability of winning at the end of the contest. Hence effort does not depend on past performance in an elimination tournament, but it does in a dynamic one. Finally, there is also related work on dynamic games, such as the war of attrition, or R&D races. However, the payoffs of these games are different from those of a tournament. The end of a war of attrition or a race occurs endogenously when all contestants but one drop out of the war, or when an innovation takes place. Yet in a tournament, the end arrives exogeneously, after a pre-specified lapse of time.

The structure of the paper is as follows. Section 2 introduces the all-pay auction with handicaps, and derives its equilibrium. Section 3 discusses several applications of the theory, including a dynamic all-pay auction, and section 4 concludes. The proofs are left to the appendix.

2 All-pay Auction with Handicaps

Consider an all-pay auction in which $N \geq 3$ risk neutral participants compete to win $K = 2$ identical prizes.[4] All of the participants in the auction must pay their bids, regardless of whether they win or lose. Hence, we can interpret their bids as the effort they put to win the contest. The prizes are allocated to the two participants with the highest effective bids $B_i = b_i + \overline{b}_i$ (randomizing to break ties, if necessary), where $b_i \geq 0$ is player i's bid, and $\overline{b}_i$ represents i's advantage in the auction. We can think of $\overline{b}_i$ as the contribution of the ability of a participant to his output. Alternatively, it can be interpreted as the realization of output (or bid) in a previous period, when the allocation of prizes is based upon total output B_i produced over two (or multiple) periods. We ellaborate on both these interpretations below.

We assume that the participants are heterogeneous. But unlike earlier models, the heterogeneity is not coming from differences in the valuations for the prizes. All participants value the prizes equally, and we denote this valuation by v. Instead, the differences among players come from their advantages. Without loss of generality, we can order the participants such that player 1 is the most advantaged, and player N is the most handicapped, and we can further normalize $\overline{b}_3 = 0$.[5] We focus here on the case of strict heterogeneity in prior advantage, so that $\overline{b}_1 > \overline{b}_2 > \overline{b}_3 = 0 > ... > \overline{b}_N$, and we will note the results that generalize when different players may share the same advantages. Assume also that $\overline{b}_1 < v$. As we will see shortly, this assumption

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[4] The results can be generalized to auctions with N participants and K prizes, with $N \geq K + 1$. We present here the simplest case, with 2 prizes to illustrate the main intuition of the results.

[5] If $\overline{b}_3 \neq 0$, we can define new advantages as $\overline{b}_i^{'} = \overline{b}_i - \overline{b}_3$. These would satisfy the normalization, and the same results would follow for the new $\overline{b}_i^{'}$.

guarantees that player 1 is active in the auction, i.e., his expected bid is strictly positive.[6] Finally, the values of v and $\left\{\bar{b}_i\right\}$ are all assumed to be common knowledge.

Given this setup, we can express player i's ex-post utility, as a function of all the players' bids, as follows:

$$u_i\left(b_1,...,b_N\right)=\begin{cases} v-b_i & \text{if } B_i>B_j \text{ for all } j\text{, except for at most one} \\ \frac{2v}{n}-b_i & \text{if } i \text{ ties with } n-1>1 \text{ others for the highest bid} \\ \frac{v}{n}-b_i & \text{if } B_j>B_i \text{ for one player } j \text{and } i \text{ ties with } n-1 \text{ others} \\ & \quad \text{for the second highest bid} \\ -b_i & \text{if } \exists j,k\neq i \text{ such that } B_i<B_j,B_k \end{cases}$$

Player i then tries to maximize his expected payoff $v\cdot\rho_i\left(b_i+\bar{b}_i\right)-b_i=v\cdot\rho_i\left(B_i\right)-B_i+\bar{b}_i$, where $\rho_i\left(B_i\right)$ is the ex-ante probability of winning, given his effective bid (and given the other players' equilibrium strategies). We will denote by $F_i\left(B_i\right)$ the cdf of player i's equilibrium strategy in the space of effective bids. Because negative bids are not allowed, it must be the case that $F_i\left(B\right)=0$ for any $B<\bar{b}_i$. Notice that ρ_i depends on $F_{-i}=\left(F_1,...,F_{i-1},F_{i+1},...,F_N\right)$, but we do not make this explicit in our formulations to ease the notation.

The following proposition characterizes the equilibrium of the all-pay auction with handicaps.

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[6] Notice that $B_i\leq v+\bar{b}_i$ for all i since no player bids above the value of the prize. Furthermore, $B_i\leq v$ since at most K players bid above v, and hence, any bid $B>v$ wins the prize with probability one. If $\bar{b}_1\geq v$, player 1 realizes that by bidding $B=\bar{b}_1$ he wins with probability one, and hence stays inactive. Then, if $\bar{b}_2<v$, the equilibrium strategies of players 2 and 3 would correspond to those of an all-pay auction with two players and one prize. If $\bar{b}_2\geq v$, all players are inactive (and 1 and 2 win with probability one). In general, with K prizes, if $\bar{b}_{K+1}=0$, any player i with $\bar{b}_i\geq v$ remains inactive.

Proposition 1. *There exists a unique Nash equilibrium of the all-pay auction with handicaps in which player i's expected payoff equals* $\max\left\{\bar{b}_i, 0\right\}$. *In this equilibrium, players* $i = 4, ..., N$ *bid* $B_i = \bar{b}_i$ *with probability one. Players* $1, 2$ *and* 3 *bid as follows:*

1. *If* $\left(\frac{v-\bar{b}_2}{v}\right) \geq \left(\frac{v-\bar{b}_1}{v}\right)^{1/2}$, *define* s *such that* $\left(\frac{v-s}{v}\right) = \left(\frac{v-\bar{b}_1}{v}\right)^{1/2}$, *and* $F(B)$ *as:*

$$F(B) = \begin{cases} \frac{\bar{b}_2}{v} & \text{if } B \in \left[0, \bar{b}_2\right] \\ \frac{B}{v} & \text{if } B \in \left[\bar{b}_2, s\right] \\ \frac{s}{v} & \text{if } B \in \left[s, \bar{b}_1\right] \\ 1 - \left(\frac{v-B}{v}\right)^{1/2} & \text{if } B \in \left[\bar{b}_1, v\right] \end{cases}$$

   Then, $F_i(B) = F(B)$ *if* $B \in \left[\bar{b}_i, v\right]$, *for* $i = 1, 2, 3$.

2. *If* $\left(\frac{v-\bar{b}_2}{v}\right) < \left(\frac{v-\bar{b}_1}{v}\right)^{1/2}$, *define* s *such that* $\left(\frac{v-\bar{b}_2}{v}\right) = \left(\frac{v-s}{v}\right)^{1/2}$, *and* $F(B)$ *as:*

$$F(B) = \begin{cases} \frac{\bar{b}_1-\bar{b}_2}{v-\bar{b}_2} & \text{if } B \in \left[\bar{b}_2, \bar{b}_1\right] \\ \frac{B-\bar{b}_2}{v-\bar{b}_2} & \text{if } B \in \left[\bar{b}_1, s\right] \\ 1 - \left(\frac{v-B}{v}\right)^{1/2} & \text{if } B \in [s, v] \end{cases}$$

   Then $F_i(B) = F(B)$ *if* $B \in \left[\bar{b}_i, v\right]$, *for* $i = 1, 2$; *but* $F_3(B) = \frac{\bar{b}_2}{v}$ *if* $B \in [0, s]$, *and* $F_3(B) = F(B)$ *if* $B \in [s, v]$.

Proof. See appendix A. □

As with other all-pay auctions with complete information, the all-pay auction with handicaps does not have a Nash equilibrium in pure strategies. Instead, all equilibria are in mixed strategies. Moreover, when players have different handicaps, the equilibrium is unique. In general, this equilibrium has the property that players $i = K+2, ..., N$ are not active in the auction. They bid zero ($B_i = \overline{b}_i$) with probability one and make zero expected profits.[7] Only the $K+1$ least handicapped players (three in this case) are active, and all bid up to $B = v$ (this is the upper support of the distribution functions of their mixed strategies). Hence, each of these players makes expected profits equal to their initial advantage $\overline{b}_i$, as bidding v wins them a prize with certainty.

Indeed, the equilibrium expected utility of the players is equal to $\max\left\{\overline{b}_i, 0\right\}$ more generally. This result also holds when the advantages are allowed to be weakly ordered (i.e., $\overline{b}_1 \geq \overline{b}_2 \geq \overline{b}_3 = 0 \geq ... \geq \overline{b}_N$).[8] It is also easy to see that the limit of the mixed strategy profiles when the advantages approach each other is also an equilibrium of the all-pay auction with equal advantages. However, other equilibria may also exist in this case, as shown by Baye et al. (1996).

Figures 1 and 2 show the shape of the equilibrium strategies of the three active players, corresponding to each of the two cases in the proposition, respectively. Consider first the case in figure 1, where the advantages (of players 1 and 2) are sufficiently dispersed. Because the cost of bidding and the value of the prize are the same for all players, those who bid in the same region use the same density function when mixing. Notice that all three players put a mass at zero (unlike the standard all-pay auction without handicaps, where at most one player does so). Above player 1's advantage, all three players' mixed strategies lie on the function $1-\sqrt{(v-B)/v}$.

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[7] Formally, player i is said to be active if $F_i\left(\overline{b}_i\right) < 1$. He therefore makes a strictly positive bid with a positive probability.

[8] It is easy to see that strict inequalities are not imposed anywhere in the proof of this result, that we provide in the appendix.

This function corresponds to the cdf of the symmetric equilibrium strategies of an all-pay auction with three players bidding for two prizes, when all participants are identical (i.e., $\overline{b}_i = 0$ for all i). As negative bids are not allowed, only players 2 and 3 bid below 1's advantage. When both players bid in this region, the second prize is always allocated to player 1. Therefore, players 2 and 3 are bidding for the remaining second prize. In that case, their strategies lie on the function B/v, that corresponds to the symmetric equilibrium of the all-pay auction with one prize and two identical players. However, in order for their mixed strategies to move to this line, neither of these two players puts any weight on the interval $\left[s, \overline{b}_1\right]$. This is in contrast to the all-pay auction without handicaps, where the support of the mixed strategies has no holes (see Baye et al., 1996) This interval exactly corresponds to the region where players 2 and 3 do not want to bid, because player 1's effective bid takes a value of $\overline{b}_1$ with strictly positive probability.[9] This region of no bidding is sustained because player 1 cannot lower his bid below his advantage. Finally, player 3 has no chance of winning a prize when bidding below player 2's advantage. As a result, this player concentrates the rest of the mass at zero.

When players 1 and 2 have similar advantages (figure 2), the previous strategies cannot be sustained, because there is no way for an increasing density function to jump from one line to the other at $\overline{b}_1$, as $\overline{b}_2/v > 1 - \sqrt{\left(v - \overline{b}_1\right)/v}$. As a result, players 1 and 2 bid more aggressively in the interval $\left[\overline{b}_1, s\right]$: they put less weight on bids $B \leq \overline{b}_1$, as their density is below $1 - \sqrt{\left(v - \overline{b}_1\right)/v}$, and this difference is compensated for by putting more weight in the interval $\left[\overline{b}_1, s\right]$. Consequently, player

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[9] We show in the appendix that whenever a player puts a mass at B, there is a δ such that no other player finds it optimal to bid in the interval $[B - \delta, B]$.

3 becomes less aggressive, putting more mass at zero, and only entering the bidding contest in the interval $[s, v]$. Again, the support of the mixed strategies of players 2 and 3 does not lie on an interval. Moreover, in the earlier case, whenever player i bids B_i, all other active players $j > i$ also bid in B_i. But, interestingly, this is not true here, as both players 1 and 2 bid in the interval $\left[\bar{b}_1, s\right]$, while player 3 does not.

3 Applications

In this section we briefly describe how these results can be applied to several settings.

3.1 Static Contests

All-pay auctions have been used to model contests. Such contests arise in lobbying, when several parties try to influence a decision; in electoral competitions, where several political parties run for election; or in organizations, where workers compete for a promotion. Most of the literature has focused on models with homogeneous participants, or participants that differ in their valuations for the prize. The later case gives some insights on the effects of heterogeneity. But this corresponds to differences in the marginal cost of effort, which do not always correspond to the differences among contestants that we see in these tournaments. In electoral competitions, one party may have a stronger electoral base. Such party would need to convince a smaller fraction of the undecided voters in order to win the election. This is better modelled with a contest where some participants are handicapped, such as ours. In organizations, a worker's output is often expressed as a liner function $y_i = e_i + a_i$, where e_i is effort and a_i is ability. When the workers differ in their ability, then a promotion tournament would correspond to an all-pay auction with handicaps as the one we study here, as well. In network industries, firms may compete to capture the market. Yet first movers may face an advantage when existing customers are reluctant to switch to a new firm. Here, again, we have another situation that can be captured by our model.

For all these cases, the result in proposition 1 can tell us something about the behavior of these contest participants (political parties, workers or firms). The following result ranks the expected bids of the players, and identifies which of them exerts more effort.

Corollary 1. *Denote by* $b_i^e = \int \left(B - \overline{b}_i\right) dF_i\left(B\right)$ *player* i*'s expected bid. Then,* $0 = b_N^e = ... = b_4^e < b_1^e < b_2^e < b_3^e$.[10]

Proof. See appendix A. □

The most handicapped players bid zero with certainty, and hence have the lowest expected bids. For the most advantaged players, their expected bid is inversely related to their ranking. Those ranked at the top have the lowest expected bid, and it is the marginal player (player 3) who spends the most resources. Hence, the relationship between a player's advantage (measured by his ex-ante ranking) and his expected bid has an inverted U-shape, the maximum being attained with player 3.[11] In contrast, when the differences are in valuations, the bids are monotonic in ranking, being highest for the player with a largest valuation.

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[10] If we allow for $\overline{b}_i \geq v$ for $i \leq K$, then the inequalities should be replaced by weak inequalities.

[11] With K prizes, this maximum is at player $K + 1$.

Notice, however, that the probability of winning a prize in the auction is still monotonic in the ranking of the player, despite the fact that the expected bid is not. For the top 3 players, the expected bid decreases as we go up the ranking. Yet, despite putting lower bids, the probability of winning is higher for the better ranked players. This is easy to verify from the fact that player i's mixed strategy first order stochastically dominates the mixed strategies of players $j > i$ (this is true for both cases in proposition 1, as can easily be seen in pictures 1 and 2). Player i therefore obtains a higher expected payoff than players $j > i$ both because it has a higher probability of winning a prize, and because, in order to do so, incurs a lower bidding cost.

3.2 Dynamic Contests

An all-pay auction with handicaps also arises naturally in dynamic contests, even when all participants are ex-ante identical. When a tournament takes place over several periods, and the final prizes are awarded on the basis of aggregate performance over these periods, the effort made in earlier periods can be captured by the advantages of our all-pay auction. Hence, the equilibrium derived above, would describe the behavior in the final period of such a tournament. Our results can then shed light on the behavior of participants in dynamic contests. Indeed, it suggests that those with an advantage in the tournament will exert low effort. Effort then increases as we go down the rankings, until it drops again for the most disadvantaged players, forming an inverted U-shape.[12]

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[12] Casas-Arce and Martínez-Jerez (2009) offer some empirical evidence for these dynamic effects in sales contests.

We can also use the results in proposition 1 to understand behavior in the earlier periods of a dynamic contest. Consider now an all-pay auction with two identical prizes, where all the players are (ex-ante) identical, and are allowed to bid for two periods. Denote these bids by b_{i1} and b_{i2}. The prizes are then allocated on the basis of the aggregate bid $B_i = \alpha \cdot b_{i1} + (1-\alpha) \cdot b_{i2}$, which is a convex combination of the two bids, with $\alpha \in (0,1)$.[13] We can think of B_i as the performance measure used to allocate prizes. This formulation allows us to consider situations in which early and late bids carry different weights. Furthermore, assume that any first-period bids are made public, so when players enter the second bidding round, there is complete information about what happend earlier. Assume also that all players value the prizes at v, as before. And assume there is no discounting. The following result describes the first-period behavior in the equilibrium of the dynamic all-pay auction, as a function of the performance measure used to allocate the prizes, summarized by α.

Proposition 2. *The equilibria of the dynamic all-pay auction satisfy the following conditions:*

1. *If $\alpha < \frac{1}{2}$, then $b_{i1} = 0$ for all i in any equilibrium.*

2. *If $\alpha = \frac{1}{2}$, then the first-period strategies are part of an equilibrium if and only if they take the following form: players i and j use any mixed (or pure) strategy with support in $[0, v]$ in the first stage, and $b_{k1} = 0$ for $k \neq i, j$.*

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[13] If $\alpha = 0$, or $\alpha = 1$, we have a single period all-pay auction, the equilibrium of which is analyzed by Baye et al. (1996).

3. *If* $\alpha > \frac{1}{2}$, *denote by* κ *the integer such that* $\kappa\left(\frac{1-\alpha}{\alpha}\right) < 1 \leq (\kappa+1)\left(\frac{1-\alpha}{\alpha}\right)$. *Then, for every* n *such that* $3 \leq n \leq N$ *there is a mixed strategy equilibrium where only* n *players are active and have bids* $b_{i1} \in \left\{0, \left(\frac{1-\alpha}{\alpha}\right)\cdot v, 2\cdot\left(\frac{1-\alpha}{\alpha}\right)\cdot v, ..., \kappa\cdot\left(\frac{1-\alpha}{\alpha}\right)\cdot v\right\}$ *with probabilities* $p_0, p_1, ..., p_\kappa$ *(the same for all players). Moreover, if* $\kappa = 1$, *there exists a unique pure strategy equilibrium where only two players are active and bid* $\left(\frac{1-\alpha}{\alpha}\right)\cdot v$, *but no pure strategy equilibrium exists for* $\kappa > 1$.

Proof. See appendix A. □

Notice that our earlier result characterizes the equilibrium payoffs in the second stage, because at this stage, the players face a static all-pay auction with handicaps. Hence, in any equilibrium of the second stage of the contest, a player's expected payoff is driven by his advantage relative to that of the player ranked third. Nevertheless, first- and second-period bids carry different weights. If we normalize the aggregate bid by $(1-\alpha)$, then $\frac{B_i}{1-\alpha} = \frac{\alpha}{1-\alpha}\cdot b_{i1} + b_{i2}$ expresses the aggregate bid in second-period bid units. We can therefore see that the second-period advantage is given by $\frac{\alpha}{1-\alpha}\cdot \max\left\{b_{i1} - b_{(3)1}, 0\right\}$.

When $\alpha < \frac{1}{2}$, the second-period bids are worth more than the first-period ones. In this case, bidding b_{i1} in the first period increases the second-period advantage by less (indeed, by at most $\frac{\alpha}{1-\alpha}\cdot b_{i1} < b_{i1}$, if $b_{(3)1} = 0$), but costs the full amount of the bid. It is therefore optimal for all players to remain inactive initially, and bid only in the second period.

When $\alpha = \frac{1}{2}$, both periods carry equal weight. Now, bidding b_{i1} increases player i's advantage by the exact same bid amount, as long as $b_{i1} \leq v$ and $b_{(3)1} = 0$. He is therefore indifferent between any bid $b_{i1} \in [0, v]$, as long as there are at most two active players in the auction. Moreover, there can be no more than two active players in equilibrium. Otherwise, $b_{(3)1} > 0$ with positive probability, and the expected second-period advantage from bidding b_{i1} is lower than the bid. Any active player would therefore receive a negative expected payoff if three or more players were active in the first period. In equilibrium, this cannot happen.

Finally, when $\alpha > \frac{1}{2}$, first-period bids weight more than second-period bids, and it becomes profitable for players to try to obtain an early advantage. We concentrate here in the set of equilibria where active players bid symmetrically (i.e., where those players with a positive expected bid use the same strategy). If α is not too large (namely, $\alpha \leq \frac{2}{3}$), a pure strategy equilibrium exists, where only two players are active and bid $b_{i1} = \left(\frac{1-\alpha}{\alpha}\right) \cdot v$. These players obtain a full advantage of v in the second period. Hence, all players are inactive in the second period. Moreover, the two players manage to preempt any further bidding in the first period. A third player can only possibly gain by bidding more than $\left(\frac{1-\alpha}{\alpha}\right) \cdot v$ at $t = 1$. Because a bid in the first-period increases the advantage at a rate of $\left(\frac{1-\alpha}{\alpha}\right) > 1$, such a player would want to increase the bid up to $\min\left\{v, 2 \cdot \left(\frac{1-\alpha}{\alpha}\right) \cdot v\right\}$. As long as $\kappa = 1$, or equivalently $\alpha \leq \frac{2}{3}$, $\min\left\{v, 2 \cdot \left(\frac{1-\alpha}{\alpha}\right) \cdot v\right\} = v$. Therefore, this third player cannot profitably gain by deviating from bidding zero.[14]

Notice that in this equilibrium, the two players are able to win the contest by bidding less than v. From the point of view of the contest design, this is not a very desirable outcome if we want to maximize the aggregate effort among all the

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[14] For $\kappa \geq 2$, on the other hand, there is a bid $b_{i1} < v$ such that this third player gains a full advantage of v in the second period. Such deviation would be profitable, and no pure strategy equilibrium exists as a result.

participants. This result suggests that it would be more desirable to have contests that give more weight to later periods, or at least, do not overweight initial periods.

Besides this pure strategy equilibrium, there are other mixed strategy equilibria where all active players bid symmetrically. It involves bids in the set $\{0, \left(\frac{1-\alpha}{\alpha}\right) \cdot v, 2 \cdot \left(\frac{1-\alpha}{\alpha}\right) \cdot v, ..., \kappa \cdot \left(\frac{1-\alpha}{\alpha}\right) \cdot v\}$. In these equilibria there are at least 3 active players, and they each make zero profits. Even though a bid in the first-period increases the advantage at a rate of $\left(\frac{1-\alpha}{\alpha}\right) > 1$, players have to trade off this benefit of a positive bid with the fact that $b_{(3)1}$ will be positive with some probability, thus reducing their profits. The probabilities of bidding each of these amounts are chosen in such a way that this trade-off is balanced, and all players are indefferent between all these bids.

Taken together, the results of proposition 2 show that it is difficult to make players bid in early rounds of a contest, unless the contest gives players explicit incentives to do so. When $\alpha \leq \frac{1}{2}$, it is weakly optimal to wait until the last period to start bidding. This is so because any advantage that players gain increases their expected future payoffs by at most the same amount it costs to produce the advantage. Therefore, players have no incentive to place early bids in order to preempt other players. Players can only successfully preempt others from bidding when $\alpha > \frac{1}{2}$, so that first-period bids are worth more than second-period bids. When that happens, if α is not too large, two players can bid early to gain a full advantage, while at the same time preventing other bidders from entering the contest. These two players would obtain a positive utility. But as α grows larger, the incentives for more players to enter the contest increases, and no successful preemption is possible.

Early bidding can become more effective if, for instance, there is asymmetric information about the ability of participants. If players know their own ability, but not the ability of others, then players could use the first period bidding as a signal of their strength. But in our model with perfect information and linear bidding costs, such incentives do not exist. Hence, the model also provides a basic framework from which to start understanding the conditions that lead to different patterns of bidding behavior in dynamic contests.

4 Conclusion

This paper studies an all-pay auction with handicaps. It derives the equilibrium of this auction, and applies it to the study of static and dynamic contests.

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5 Appendix A

In order to proof the main proposition of the paper, we will first proceed in steps through different lemmas. We derive most of the results for general all-pay auctions with handicaps, with N participants and K prizes. We turn to the special case of $K = 2$ when deriving the formulas for the strategies of the players.

Denote by I_i the support of F_i. First notice that $F_i(B) = 0$ for any $B < \bar{b}_i$, as bids b_i are non-negative. Moreover, $F_i(B) = 1$ for any $B \geq v + \bar{b}_i$, as nobody bids above the value of the prize. Furthermore, $F_i(B) = 1$ for any $B \geq v$ because at most K players bid above v, and hence, any bid $B > v$ wins the prize with probability one. Let B_i^u be the upper bound of player i's support I_i: $F_i(B_i^u) = 1$ and $F_i(B) < 1$ for any $B < B_i^u$. We say that player i is active if $B_i^u > \bar{b}_i$. Likewise, let B_i^l be the lower bound of I_i, which satisfies $B_i^l \geq \bar{b}_i$.

Lemma 1. *F_i is continuous for $B_i > \bar{b}_i$.*

Proof. We now show that the equilibrium distribution function for player i is continuous above zero (for $B' > \bar{b}_i$). Suppose that player i puts a mass on $B' > \bar{b}_i$, and hence, F_i is discontinuous at that point. Notice that $\rho_i(B') > 0$, otherwise bidding B' would not be optimal ($B = \bar{b}_i$ would lower the payment while keeping the probability of winning constant). Consider now any player $j \neq i$ such that $B' \geq \bar{b}_j$. It follows that $\rho_j(B') < \lim_{B \searrow B'} \rho_j(B)$ (where $\lim_{B \searrow B'}$ denotes the limit from above), and hence $v \cdot \rho_j(B) - B + \bar{b}_j$ is discontinuous and jumps upwards after B'. Therefore, player j prefers to bid $B' + \varepsilon$, rather than bidding on the interval $[B' - \delta, B']$, where $\varepsilon, \delta > 0$ are small. As this is true for all $j \neq i$, nobody bids on the interval $[B' - \delta, B']$, except player i. Hence, $\rho_i(B') = \rho_i(B' - \delta)$, and putting a mass at B' cannot be optimal, as bidding B' is dominated by bidding $B' - \delta$. If follows that F_i must be continuous at any $B' > \bar{b}_i$. Moreover, notice that $F_i\left(\bar{b}_i\right) = \lim_{B \searrow \bar{b}_i} F_i(B)$. $\square$

Next, we show that $K + 1$ players share the same upper bound of the support of their mixed strategies

Lemma 2. *Let* $\overline{B} = \max_i \{B_i^u\}$. *Then* $B_i^u = \overline{B}$ *for at least* $K + 1$ *players.*

Proof. Suppose, otherwise, that $B_i^u = \overline{B}$ for at most K players, indexed $\{i_1, ..., i_k\}$ $(k \leq K)$, and $B_i^u < \overline{B}$ for the rest. Let $\overline{B}' = \max\left\{B_i^u, s.t.\ B_i^u < \overline{B}\right\} < \overline{B}$. Then, for players $i \in \{i_1, ..., i_k\}$, $\rho_i(B) = 1$ for any $B > \overline{B}'$. Therefore, it cannot be optimal to bid up to $\overline{B}$, as a bid of $\overline{B}' + \varepsilon < \overline{B}$ (for ε small enough) wins the prize with certainty. $\square$

The next lemma will be useful to show the first part of the proposition.

Lemma 3. *Suppose there exists* $i \leq K + 1$ *such that* $B_i^l = \bar{b}_i$. *Then,* $\overline{B} = v$.

Proof. Assume, otherwise, that $\overline{B} < v$. Denote by j, the player with the largest index among the players $i \leq K + 1$ such that $B_i^l = \bar{b}_i$. From lemma 1, all players i with $j < i \leq K + 1$ (if there is any) bid continuously and cannot put a mass anywhere, because their mixed strategy has $B_i^l > \bar{b}_i$. Moreover, these players must have $B_i^l \geq \bar{b}_j$ (otherwise the player with the lowest B_i^l among those would make negative profits). Therefore, player j must make zero profits when bidding $B = \bar{b}_j$.

But this cannot be an equilibrium if $\overline{B} < v$, because this player could make positive profits by bidding $B = v - \varepsilon$, for $\varepsilon < v - \overline{B}$. $\square$

This lemma characterizes the upper bound of the densities.

Lemma 4. *$\overline{B} = v$.*

Proof. Suppose, instead, that $\overline{B} < v$. From lemma 3, it must be the case that $B_i^l > \bar{b}_i$ for all $i \leq K + 1$. Therefore, all players bid countinuously in $(0, v]$. Let $\underline{B} = \min \left\{ B_i^l, i \leq K + 1 \right\} > 0$, and let j be a player such that $B_j^l = \underline{B}$. As there are at least K players bidding strictly above $\underline{B}$ with probability one, player j loses with certainty when bidding $\underline{B}$. He makes negative profits of $-\underline{B} + \bar{b}_j$ in this case, which cannot be possible in equilibrium. $\square$

It is now possible to derive the expected payoffs of all the participants:

Lemma 5. *The expected payoff of player i is* $\max \left\{ \bar{b}_i, 0 \right\}$.

Proof. Because $K + 1$ players bid up to v, it must be the least handicapped ones, as any $i > K + 1$ would make negative profits when bidding v. From lemma 4, it then follows directly that players $i \leq K + 1$ must earn profits of $\bar{b}_i$.

Players $i > K + 1$ must make zero profits. To see this, suppose that player $i > K + 1$ makes positive profits, instead. As $\bar{b}_i \leq 0$, it must be the case that $B_i^u < v$ and $\rho_i \left(B_i^u \right) > 0$. Notice, however, that $\rho_i \left(B_i^u \right) \leq \rho_{K+1} \left(B_i^u \right)$. Therefore, player $K + 1$, when bidding B_i^u, must earn $v \cdot \rho_{K+1} \left(B_i^u \right) - B_i^u \geq v \cdot \rho_i \left(B_i^u \right) - \left(B_i^u - \bar{b}_i \right) > 0$. But this cannot be the case, because player $K + 1$ makes zero profits in equilibrium. $\square$

Proof of Proposition 1. Consider the player with the highest upper support among players $j > K+1$, and denote him by l. If this player was active, $B_l^u \geq \bar{b}_K$ as otherwise he loses with probability one and would rather not be active. Furthermore, $v \cdot \rho_l\left(B_l^u\right) - B_l^u + \bar{b}_l \geq 0$, which implies that $\rho_l\left(B_l^u\right) \geq \frac{B_l^u - \bar{b}_l}{v} > \frac{B_l^u}{v}$. Consider now player $K+1$. If he bids B_l^u, he wins with probability $\rho_{K+1}\left(B_l^u\right) \geq \rho_l\left(B_l^u\right) > \frac{B_l^u}{v}$, and would make strictly positive profits.[15] From the previous paragraph, this cannot happen, and hence, player l cannot be active. In turn, no player $j > K+1$ is active.

As a result, player i is active and has upper bound $B_i^u = v$ if and only if $i \leq K+1$. From now on, we will only consider the active players.

Suppose that $B \in I_i$ and $B \in I_j$. Then, $\rho_i(B) = \rho_j(B) = \frac{B}{v}$. Furthermore, because there are only $K+1$ active players, the probability of winning for any of them takes a simple form: $\rho_i(B) = 1 - \prod_{l \neq i}\left(1 - F_l(B)\right)$. Then, $1 - \prod_{l \neq i}\left(1 - F_l(B)\right) = 1 - \prod_{l \neq j}\left(1 - F_l(B)\right)$, which implies that $F_i(B) = F_j(B)$.

Consider now two players i and j, such that $F_i\left(\widehat{B}\right) < F_j\left(\widehat{B}\right)$ for some $\widehat{B} > \bar{b}_i, \bar{b}_j$. Denote by $\widehat{B}'$ the point such that $F_j\left(\widehat{B}'\right) = F_j\left(\widehat{B}\right)$, but $F_j(B) < F_j\left(\widehat{B}\right)$ for any $B < \widehat{B}'$. As $\widehat{B}' \in I_j$ it must be the case that $\rho_j\left(\widehat{B}'\right) = \frac{\widehat{B}'}{v}$, and so $\prod_{l \neq j}\left(1 - F_l\left(\widehat{B}'\right)\right) = \frac{v - \widehat{B}'}{v}$. If $\widehat{B}' \geq \bar{b}_i$, then it must also be the case that $\rho_i\left(\widehat{B}'\right) \leq \frac{\widehat{B}'}{v}$, which implies that $\prod_{l \neq i}\left(1 - F_l\left(\widehat{B}'\right)\right) = \frac{\left(1 - F_j\left(\widehat{B}'\right)\right)\left(\frac{v - \widehat{B}'}{v}\right)}{\left(1 - F_i\left(\widehat{B}'\right)\right)} \geq \frac{v - \widehat{B}'}{v}$. The inequality is satisfied as long as $F_i\left(\widehat{B}'\right) \geq F_j\left(\widehat{B}'\right)$. However, this cannot happen because $F_i(B) < F_j(B)$ for any $B \in \left(\widehat{B}', \widehat{B}\right)$ and $F_j\left(\widehat{B}'\right) = \lim_{B \searrow \widehat{B}'} F_j(B) = F_j\left(\widehat{B}\right) > F_i\left(\widehat{B}'\right)$. As a result, player j cannot bid on the interval $\left[\bar{b}_i, \widehat{B}\right]$. Moreover, as

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[15] $\rho_{K+1}\left(B_l^u\right)$ equals the probability that at least one player among $i \leq K$ bids below B_l^u, while $\rho_l\left(B_l^u\right)$ is the probability that at least two players among $i \leq K+1$ bid below B_l^u (and therefore, at least one player among $i \leq K$ bids below B_l^u). Since the second condition is more stringent, it follows immediately that $\rho_{K+1}\left(B_l^u\right) \geq \rho_l\left(B_l^u\right)$.

$F_j\left(\bar{b}_i\right) = F_j\left(\widehat{B}\right) > 0$, it must be the case that j is more disadvantaged than i $(j > i)$. Therefore, $F_i\left(B\right) \leq F_j\left(B\right)$ for all $B \in [0, v]$, whenever $j > i$. Furthermore, if $F_i\left(B\right) = F_j\left(B\right)$ for some $B \in \left[\bar{b}_i, v\right]$, then $F_i\left(B'\right) = F_j\left(B'\right)$ for any $B' \geq B$ (if $F_i\left(B'\right) < F_j\left(B'\right)$ for some B' we would have $F_i\left(B\right) < F_j\left(B\right)$ as well).

Finally, we can see that $B_i^l = \bar{b}_i$ for all $i \leq K+1$, so that $\bar{b}_i \in I_i$. Suppose, otherwise, that $B_i^l > \bar{b}_i$ for player i. Because player i cannot put a mass at B_i^l, it must be that $F_i\left(B_i^l\right) = 0$. Then, $F_j\left(B_i^l\right) = 0$ for all $j \leq i$. Furthermore, if $F_j\left(B_i^l\right) > 0$ for some $j > i$, then $F_j\left(\bar{b}_i\right) = F_j\left(B_i^l\right)$. It follows that $\rho_i\left(B_i^l\right) = 1 - \prod_{l \neq i}\left(1 - F_l\left(B_i^l\right)\right) = 1 - \prod_{l \neq i}\left(1 - F_l\left(\bar{b}_i\right)\right) = \rho_i\left(\bar{b}_i\right)$. But this is a contradiction, as $\bar{b}_i$ would yield a higher payoff than B_i^l.

We can now construct the equilibrium. We focus on the case $K = 2$ from now on, but an analogous reasoning would yield the equilibrium for any K.

Because player 2 bids $B = \bar{b}_2$, it must be the case that $\rho_2\left(\bar{b}_2\right) = F_3\left(\bar{b}_2\right) = \frac{\bar{b}_2}{v}$. Player 3 then must bid zero with probability $\frac{\bar{b}_2}{v}$ (because nobody else bids on the interval $\left(0, \bar{b}_2\right)$). Player 1 also bids $B = \bar{b}_1$, and therefore $\rho_1\left(\bar{b}_1\right) = 1 - \left(1 - F_2\left(\bar{b}_1\right)\right)\left(1 - F_3\left(\bar{b}_1\right)\right) = \frac{\bar{b}_1}{v}$. Here we need to distinguish two cases. If $\left(\frac{v - \bar{b}_2}{v}\right) < \left(\frac{v - \bar{b}_1}{v}\right)^{1/2}$, then in order to satisfy $\rho_1\left(\bar{b}_1\right) = \frac{\bar{b}_1}{v}$, it must be the case that $F_2\left(\bar{b}_1\right) < \frac{\bar{b}_1}{v}$ when $F_3\left(\bar{b}_1\right) = \frac{\bar{b}_1}{v}$. As a result, neither player 2 nor 3 can bid in the interval $\left(\bar{b}_2, \bar{b}_1\right)$, because otherwise $F_2\left(\bar{b}_1\right) = F_3\left(\bar{b}_1\right) > \frac{\bar{b}_1}{v}$, and the condition would never be satisfied. Player 2, then, puts enough probability on $B = \bar{b}_2$, so that $\rho_1\left(\bar{b}_1\right) = \frac{\bar{b}_1}{v}$, which yields $F_2\left(\bar{b}_2\right) = F_2\left(\bar{b}_1\right) = \frac{\bar{b}_1 - \bar{b}_2}{v - \bar{b}_2}$. As $F_2\left(\bar{b}_1\right) < F_3\left(\bar{b}_1\right)$, player 3 is not bidding on any B such that $F_2\left(B\right) < \frac{\bar{b}_2}{v}$. Denote by s the bid such that $F_2\left(s\right) = \frac{\bar{b}_2}{v}$. Players 1 and 2 must be bidding on the same support on $\left(\bar{b}_1, s\right)$, and hence, by continuity $F_1\left(B\right) = F_2\left(B\right)$

for any $B \in \left(\bar{b}_1, s\right)$. If $B \in I_1 \cap \left(\bar{b}_1, s\right)$, then

$$\begin{aligned} \rho_1(B) &= 1 - (1 - F_1(B))(1 - F_3(B)) \\ &= 1 - (1 - F_1(B))\left(\frac{v - \bar{b}_1}{v}\right) = \frac{B}{v} \end{aligned}$$

which yields $F_1(B) = \frac{B - \bar{b}_2}{v - \bar{b}_2}$. Because $F_1(B)$ is continuous, the entire interval $\left[\bar{b}_1, s\right]$ must be part of the support for the two players, and s satisfies $\left(\frac{v - \bar{b}_2}{v}\right) = \left(\frac{v-s}{v}\right)^{1/2}$. Furthermore, $F_1(B) = F_2(B) = F_3(B)$ for any $B \in [s, v]$. If $B \in I_1 \cap [s, v]$, then

$$\rho_1(B) = 1 - (1 - F_1(B))^2 = \frac{B}{v}$$

which yields $F_1(B) = 1 - \left(\frac{v-B}{v}\right)^{1/2}$. Because $F_1(B)$ is continuous and $F_1(v) = 1$, the entire interval $[s, v]$ must be part of the support for the three players.

Alternatively, if $\left(\frac{v - \bar{b}_2}{v}\right) \geq \left(\frac{v - \bar{b}_1}{v}\right)^{1/2}$, both players 2 and 3 must bid on the interval $\left(\bar{b}_2, \bar{b}_1\right)$ in order to have $\rho_1\left(\bar{b}_1\right) = \frac{\bar{b}_1}{v}$. Hence $F_2(B) = F_3(B)$ for all $B \geq \bar{b}_2$. When bidding in this interval, player 1 wins with probability one, and $\rho_2(B) = F_3(B) = \frac{B}{v} \left(= F_2(B)\right)$. By continuity, they must bid on the interval $\left[\bar{b}_2, s\right]$, where s satisfies $\left(\frac{v-s}{v}\right) = \left(\frac{v - \bar{b}_1}{v}\right)^{1/2}$. Then, for $F_2(s) = F_2\left(\bar{b}_1\right) = F_3\left(\bar{b}_1\right) = \frac{s}{v}$, the condition $\rho_1\left(\bar{b}_1\right) = \frac{\bar{b}_1}{v}$ holds. Finally, in the interval $\left[\bar{b}_1, v\right]$, players 2 and 3 have the same support. Because player 1 cannot bid alone, he must too have the same support. Consequently, $F_1(B) = F_2(B) = F_3(B)$ for any $B \in \left[\bar{b}_1, v\right]$. As in the previous case, if $B \in I_1 \cap \left[\bar{b}_1, v\right]$, then

$$\rho_1(B) = 1 - (1 - F_1(B))^2 = \frac{B}{v}$$

which yields $F_1(B) = 1 - \left(\frac{v-B}{v}\right)^{1/2}$. Because $F_1(B)$ is continuos and $F_1(v) = 1$, the entire interval $\left[\bar{b}_1, v\right]$ must be part of the support for the three players. $\square$

Proof of Corollary 1. It suffices to show that $b_3^e > b_2^e > b_1^e > 0$. Notice first that $F_2(B) = F_1(B) = F(B)$ for all $B \in \left[\bar{b}_1, v\right]$. Then,

$$\begin{aligned} b_2^e &= \int_{\bar{b}_2}^{v} \left(B - \bar{b}_2\right) dF_2(B) = \int_{\bar{b}_2}^{\bar{b}_1} \left(B - \bar{b}_2\right) dF_2(B) + \int_{\bar{b}_1}^{v} \left(B - \bar{b}_2\right) dF_2(B) \\ &\geq \int_{\bar{b}_1}^{v} \left(B - \bar{b}_2\right) dF_2(B) > \int_{\bar{b}_1}^{v} \left(B - \bar{b}_1\right) dF_1(B) = b_1^e \end{aligned}$$

The same argument yields $b_2^e < b_3^e$ under case 1 in proposition 1, as $F_2(B) = F_3(B)$ for all $B \in \left[\bar{b}_2, v\right]$. For case 2, $F_2(B) = F_3(B)$ only for $B \in [s, v]$. Still, we have

$$\begin{aligned} b_2^e &= \int_{\bar{b}_2}^{s} BdF_2(B) + \int_{s}^{v} BdF_2(B) - \bar{b}_2 < \bar{b}_2 \cdot \left(\frac{s}{v} - 1\right) + \int_{s}^{v} BdF_2(B) \\ &< \int_{s}^{v} BdF_2(B) = \int_{s}^{v} BdF_3(B) = \int_{0}^{v} BdF_3(B) = b_3^e \end{aligned}$$

where the first inequality follows from the fact that $\int_{\bar{b}_2}^{s} BdF_2(B) < s \cdot \frac{\bar{b}_2}{v}$, because the right hand side amounts to putting the entire mass of $F_2(s) = \frac{\bar{b}_2}{v}$ at s. $\square$

Proof of Proposition 2. First, notice that allocating prizes based on $B_i = \alpha \cdot b_{i1} + (1-\alpha) \cdot b_{i2}$ is equivalent to doing so based on $B_i' = \frac{\alpha}{1-\alpha} \cdot b_{i1} + b_{i2}$. Therefore, in the second period, we have an all-pay auction with handicaps given by $\left\{\frac{\alpha}{1-\alpha} \cdot b_{i1}\right\}_{i=1}^{i=N}$. From Proposition 1, we know that the expected second-period payoff for player i is given by $u_{i2}^e(b_{i1}) = E\left[\max\left\{0, \min\left\{v, \frac{\alpha}{1-\alpha} \cdot \left(b_{i1} - b_{(3)1}\right)\right\}\right\}\right]$, where $b_{(3)1}$ is the third order statistic of the first-period equilibrium bids, and the expectation is taken over the distribution of $b_{(3)1}$ given the equilibrium strategies of the other players. Therefore, the first-period utility of bidding b_{i1} is $u_{i1}\left(b_{i1}\right) = u_{i2}^{e}\left(b_{i1}\right) - b_{i1}$.

When $\alpha < \frac{1}{2}$, $u_{i1}\left(b_{i1}\right) \leq \frac{\alpha}{1-\alpha} \cdot b_{i1} - b_{i1} < 0$ for any $b_{i1} \in (0, v]$. Therefore, it is optimal for all players to bid 0 in the first period.[16]

When $\alpha = \frac{1}{2}$, $u_{i2}^{e}\left(b_{i1}\right) \leq b_{i1}$, with equality if and only if $b_{(3)1} = 0$ with probability one. Therefore, there are at most two active players. Otherwise, $E\left[b_{(3)1}\right] > 0$, and all active players would be better off bidding 0, since $u_{i1}\left(b_{i1}\right) < 0$ for all $b_{i1} \in (0, v]$.

For these two active players, $u_{i1}\left(b_{i1}\right) = 0$ for all b_{i1}, and therefore, they are indifferent among any bid $b_{i1} \in [0, v]$, and any strategy on this interval can be supported as an equilibrium.

Suppose now that $\alpha > \frac{1}{2}$. Denote by $b_{(2)1}^{i}$ the second order statistic of the first period equilibrium bids of all the players except i. Notice that $b_{(3)1} = b_{(2)1}^{i}$ whenever $b_{i1} > b_{(2)1}^{i}$, and $u_{i2}^{e} = 0$ otherwise. Denote by $F_{(2)1}^{i}$ the distribution of $b_{(2)1}^{i}$.

Let I_{i1} be the support of player i's strategy F_{i1}, and $I_{(3)1}$ the support of the distribution of $b_{(3)1}$. Let $\underline{B}_{i1} \equiv \inf I_{i1}$, and $\underline{B}_{(3)1} \equiv \inf I_{(3)1}$. First, notice that $\underline{B}_{(3)1} = 0$. Otherwise, if $\underline{B}_{(3)1} > 0$, a player that bids $b_{i1} = \underline{B}_{(3)1}$ obtains a negative payoff of $u_{i1}\left(b_{i1}\right) = -\underline{B}_{(3)1}$. As a result, since the strategies are independent across players, there are at most 2 players for whom $\underline{B}_{i1} > 0$, and hence, no more than 2 players can make positive profits, while the rest must make zero profits.

If there are only two active players, $b_{(3)1} = 0$ with probability 1. The two active players would then bid $b_{i1} = \left(\frac{1-\alpha}{\alpha}\right) v$, since for them u_{i1} is increasing until this point,

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[16] Notice that, for all values of α, no players bids $b_{i1} > v$.

and decreasing thereafter. For this to be an equilibrium, it cannot be optimal for a third player j to place a positive bid. The utility of bidding $b_{j1} \in \left(0, \left(\frac{1-\alpha}{\alpha}\right) v\right]$ would be negative since $b_{(3)1} = b_{j1}$ in this case. If $b_{j1} > \left(\frac{1-\alpha}{\alpha}\right) v$, then $b_{(3)1} = \left(\frac{1-\alpha}{\alpha}\right) v$, and $u_{i1}\left(b_{i1}\right) = \frac{\alpha}{1-\alpha} \cdot \left(b_{i1} - \left(\frac{1-\alpha}{\alpha}\right) v\right) - b_{i1}$. It is therefore optimal for j to increase the bid up to $\min\left\{v, 2\left(\frac{1-\alpha}{\alpha}\right) v\right\}$. If $v > 2\left(\frac{1-\alpha}{\alpha}\right) v$, bidding $b_{j1} = 2\left(\frac{1-\alpha}{\alpha}\right) v$ would yield j profits of $u_{i1}\left(b_{i1}\right) = v - 2\left(\frac{1-\alpha}{\alpha}\right) v > 0$, so the equilibrium with only two active players cannot be sustained. Instead, when $v \leq 2\left(\frac{1-\alpha}{\alpha}\right) v$, j makes non-positive profits when bidding $b_{j1} = v$, so no player would like to deviate. Hence, the second part of case 3 of the proposition follows.

Suppose now that there are $n \geq 3$ active players. In this case, if there is a pure strategy equilibrium, $b_{(3)1}$ would be constant and positive. The player that bids $b_{(3)1}$ would make negative profits, and therefore, he would rather stay inactive. Hence, with three or more active players, the equilibrium must be in mixed strategies.

Suppose that player i bids $b_{i1} \in \left(0, \left(\frac{1-\alpha}{\alpha}\right) v\right)$. Then, no other player can bid in $\left[b_{i1}, \left(\frac{1-\alpha}{\alpha}\right) v\right)$. To see this, notice that $0 \leq u_{i1}\left(b_{i1}\right) = \left(\frac{\alpha}{1-\alpha}\right) \cdot b_{i1} \cdot \Pr\left(b^{i}_{(2)1} \leq b_{i1}\right) - \left(\frac{\alpha}{1-\alpha}\right) \cdot \int_{0}^{b_{i1}} b dF^{i}_{(2)1}(b) - b_{i1} \leq \left(\frac{\alpha}{1-\alpha}\right) \cdot b_{i1} \cdot \Pr\left(b^{i}_{(2)1} \leq b_{i1}\right) - b_{i1}$, and therefore, $\Pr\left(b^{i}_{(2)1} \leq b_{i1}\right) \geq \frac{1-\alpha}{\alpha}$. However, we cannot have $\Pr\left(b^{i}_{(2)1} \leq b_{i1}\right) > \frac{1-\alpha}{\alpha}$, as otherwise, u_{i1} would be increasing at b_{i1}, and that bid would not be optimal. To see this, let $x \in \left[b_{i1}, \left(\frac{1-\alpha}{\alpha}\right) v\right]$. Then, $\Delta u_{i1} = u_{i1}(x) - u_{i1}\left(b_{i1}\right) \geq$

$u_{i1}(x) = \left(\frac{\alpha}{1-\alpha}\right) \cdot x \cdot \Pr\left(b^{i}_{(2)1} \leq b_{i1}\right) - \left(\frac{\alpha}{1-\alpha}\right) \cdot \int_{0}^{b_{i1}} b dF^{i}_{(2)1}(b) + \left(\frac{\alpha}{1-\alpha}\right) \cdot \int_{b_{i1}}^{x} (x-b)\, dF^{i}_{(2)1}(b) - x$. Since $\Pr\left(b^{i}_{(2)1} \leq b_{i1}\right) \geq \frac{1-\alpha}{\alpha}$, $\left(\frac{\alpha}{1-\alpha}\right) \cdot x \cdot \Pr\left(b^{i}_{(2)1} \leq b_{i1}\right) - x$ is (weakly) increasing in x. Moreover, when another player bids in $\left[b_{i1}, \left(\frac{1-\alpha}{\alpha}\right) v\right)$, we must have $\int_{b_{i1}}^{x} (x-b)\, dF^{i}_{(2)1}(b) > 0$ for $x = \left(\frac{1-\alpha}{\alpha}\right) v$, and hence $u_{i1}\left(\left(\frac{1-\alpha}{\alpha}\right) v\right) > u_{i1}\left(b_{i1}\right)$, which would contradict the fact that $b_{i1} \in I_{i1}$.

It immediately follows that there is at most one player bidding in $\left(0, \left(\frac{1-\alpha}{\alpha}\right) v\right)$. From the previous result, if i makes the lowest bid in this interval, no other player can bid above him, and therefore, it is only i bidding in the entire interval. In a symmetric equilibrium, all active players must therefore bid in $\{0\} \cup \left[\left(\frac{1-\alpha}{\alpha}\right) v, v\right]$. Iterating this argument, it follows that in a symmetric equilibrium no player bids in the intervals $\left(k \left(\frac{1-\alpha}{\alpha}\right) v, (k+1) \left(\frac{1-\alpha}{\alpha}\right) v\right)$, for $k = 0, ...\kappa$. $\square$

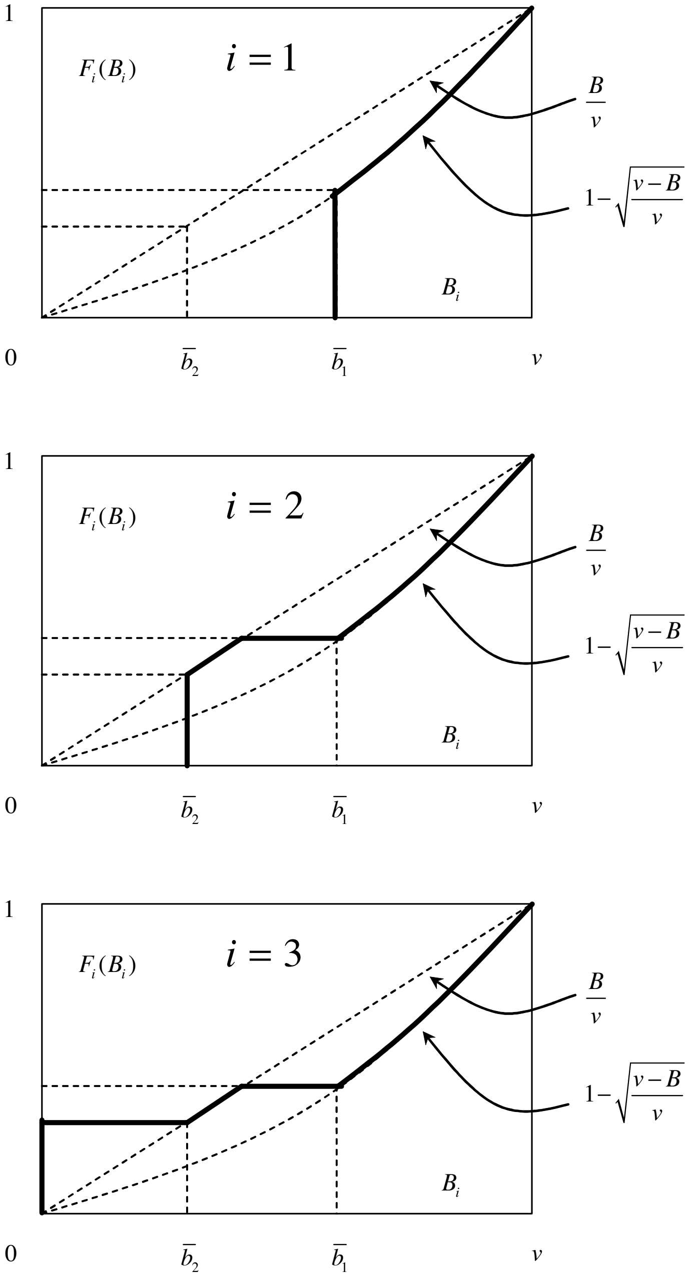
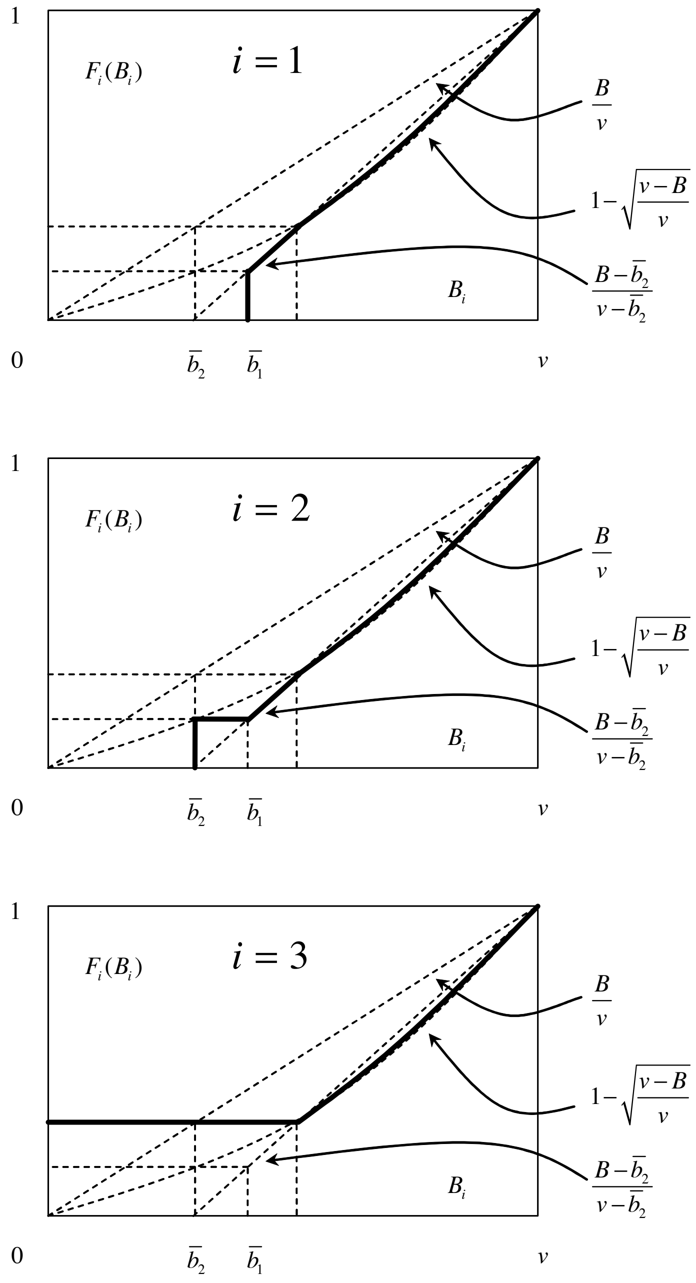

1
$F_i(B_i)$
$i = 1$
$\frac{B}{v}$
$1-\sqrt{\frac{v-B}{v}}$
B_i
0
$\bar{b}_2$
$\bar{b}_1$
v
1
$F_i(B_i)$
$i = 2$
$\frac{B}{v}$
$1-\sqrt{\frac{v-B}{v}}$
B_i
0
$\bar{b}_2$
$\bar{b}_1$
v
1
$F_i(B_i)$
$i = 3$
$\frac{B}{v}$
$1-\sqrt{\frac{v-B}{v}}$
B_i
0
$\bar{b}_2$
$\bar{b}_1$
v

Figure 1:

Figure 2:


1
$F_i(B_i)$
$i = 1$
$\frac{B}{v}$
$1-\sqrt{\frac{v-B}{v}}$
$\frac{B-\overline{b}_2}{v-\overline{b}_2}$
B_i
0
$\overline{b}_2$
$\overline{b}_1$
v
1
$F_i(B_i)$
$i = 2$
$\frac{B}{v}$
$1-\sqrt{\frac{v-B}{v}}$
$\frac{B-\overline{b}_2}{v-\overline{b}_2}$
B_i
0
$\overline{b}_2$
$\overline{b}_1$
v
1
$F_i(B_i)$
$i = 3$
$\frac{B}{v}$
$1-\sqrt{\frac{v-B}{v}}$
$\frac{B-\overline{b}_2}{v-\overline{b}_2}$
B_i
0
$\overline{b}_2$
$\overline{b}_1$
v