

# Tournament for Belief and Optimal Uncertainty

Illoong Kwon

Seoul National University

February 2011

(Very Preliminary)

(Comments are Welcome.)

## Abstract

This paper considers a promotion tournament where the winner is decided by the principal's posterior belief about the agents' ability, not necessarily by the agents' performance, and analyzes the potential tradeoff between incentive effect and sorting effect of promotion tournaments. Contrary to the previous studies, this paper shows that the agents' optimal choice of uncertainty is finite. If the uncertainty is infinite, the agent's performance or effort would have no effect on the principal's posterior belief. Therefore, promotion decisions based on a manager's subjective performance rating can reduce the choice of risky strategies by the agents. This paper also shows that the principal's optimal choice of uncertainty is strictly positive and finite as well.

## 1 Introduction

Tournament theory, à la Lazear and Rosen (1981), has been one of the key theoretical building blocks in the analysis of personnel policies within firms, especially in the analysis of promotions. In a standard tournament model, players' (or agents') performance depends on their effort and random shocks. Then, a player with the highest realized performance wins the tournament and receives a fixed prize. The literature has extended the standard model in various directions

including multi-stage tournament, asymmetric tournament, and repeated tournaments, but the basic structure of the model where a player with relatively high *performance* wins the prize has remained the same.

However, there are two aspects of the standard tournament model that differ from the actual practice of promotions. First, in real-life promotions, a supervisor (or a principal) observes the workers' performance for many periods and promotes the one who is believed to have the highest ability for the higher-ranked job. That is, it is not the realized performance of the players but the supervisor's subjective *belief* about the players' ability that determines the winner. Second, even though the standard tournament model has largely focused on the incentive effect of the tournament, promotions also provide a sorting effect. That is, promotion tournaments not only drive workers to exert more *effort*, but also help the firm to select the highest *ability* workers for the higher-ranked job.

Therefore, this paper departs from the standard tournament model by choosing the winner of a tournament based on the principal's (or supervisor's) posterior belief on the agents' ability, not on an objective performance measure. Of course, if the principal believes that the agent with the highest performance is most likely to have the highest ability, then there would be no practical difference between the standard tournament model based on an objective performance measure and the model based on the principal's subjective belief. Also, there would be no trade-off between the incentive effect and the sorting effect of promotions.

This paper shows, however, that the tournaments based on objective performance and those based on subjective belief are no longer equivalent if the risk or uncertainty of the agents' performance is endogenous. For example, suppose that one agent has chosen a risky strategy where his performance largely depends on random shocks, or luck, and that the other agent has chosen a safe strategy where his performance largely depends on ability and effort. Then, even if the former agent has higher realized performance than the latter, the principal may still promote the latter because her posterior belief about the latter agent's ability can be higher than her belief about the former agent's.

The importance of promotions as a sorting device reflects potentially large heterogeneity in

players' ability. This paper also shows that with potential heterogeneity in players' ability, there is a trade-off between the sorting effect and the incentive effect of promotions. That is, while the principal would like to minimize the uncertainty in the players' performance measure for efficient sorting, a precise performance measure can reveal the difference in players' ability too soon and reduce the players' incentives for effort. Therefore, the principal may intentionally choose a noisy performance measure.

To my best knowledge, this paper is the first to analyze a tournament where the winner is decided by the principal's subjective belief about the agents' ability. As discussed above, this model can make a meaningful contribution only if the risk or uncertainty of the agents' performance (or performance measure) is endogenous, and there are a small number of studies that consider endogenous risk-taking in a standard tournament model. For example, Hvide (2002) analyzes a tournament where the agents can choose their risk as well as their effort, and shows that the agents will choose infinite risk, if possible, which results in zero effort.<sup>1</sup> This result raises serious alarms as many agents, for example fund managers, have much discretion in choosing the risk of their performance (Chevalier and Ellison 1997). In contrast, this paper shows that the agents do not take an infinite amount of risk, if the winner of a tournament is decided based on the principal's subjective belief about the agents' ability.

There are relatively many studies that have explored the idea that a principal may intentionally withhold information on the agents' mid-term performance in a dynamic model. Some recent studies include Ederer (2009), Gershkov and Perry (2006), Goltsman and Mukherjee (2006), Yildirim (2005), and Aoyagi (2010). However, in these models, the principal privately observes the agents' mid-term performance. Therefore, the decision to release the mid-term performance information does not change the principal's belief about the agents' ability, or the sorting effect of the tournament. Consequently, these models are largely interested in maximizing the agents' effort (i.e. incentive effect), not in selecting the highest ability agent (i.e. sorting effect).<sup>2</sup>

---

<sup>1</sup>In a non-tournament setting (e.g. with only one agent), there is a series of studies that have studied the optimal contract when an agent can choose both his risk and his effort. See, for example, Diamon (1998), Hellwig (1994), Biais and Casamatta (1999), and Palomino and Prat (2003).

<sup>2</sup>There exists a related recent literature on sequential auctions and elimination contests that focus on the ef-

The rest of the paper is organized as follows. Section 2 sets up the basic model. Section 3 considers a case where the principal can choose the risk or uncertainty in the agents' performance. In section 4, the case where the agents can choose their own risk is analyzed. Section 5 summarizes and concludes.

## 2 Model

I consider a tournament with two risk-neutral players ( $i = A, B$ ). The principal observes the agents' performance for two periods ( $t = 1, 2$ ), and updates her belief on each player's ability. Then, at the end of the second period, the principal promotes an agent with higher expected ability. Therefore, as discussed in the beginning, I consider a typical promotion setting where the winner is decided by a supervisor's subjective belief.

More specifically, in the first period, each player's performance ( $y_{i1}$ ) is determined by his ability  $\eta_i$  and a random shock  $\epsilon_{i1}$  such that

$$y_{i1} = \eta_i + \epsilon_{i1}.$$

For simplicity, I assume that the first period performance does not depend on the player's effort. One can consider the first period performance as a signal for each player's ability.

The first period performance is public information, but neither the players nor the principal observe  $\eta_i$ . The prior distribution of  $\eta_i$  is normal with mean  $m_0$  and variance  $\sigma_0^2$ . The noise term  $\epsilon_{i1}$  follows a normal distribution with mean zero and variance  $\sigma_{\epsilon_i}^2$ .

In the second period, each player exerts effort  $a_i$  and the performance  $y_{i2}$  is determined as follows:

$$y_{i2} = \eta_i + a_i + \epsilon_{i2},$$

where  $\epsilon_{i2}$  follows a normal distribution with mean zero and variance  $\sigma_{\epsilon_i}^2$ . The cost of effort is  $g(a) = e^a - 1$ .  $\epsilon_{it}$  are independent across  $i$  and  $t$  ( $i = A, B, t = 1, 2$ ).

---

iciency of the sorting effects. See, for example, Moldovanu and Sela (2006), Mezzetti et al. (2008), Cai et al. (2007), and Zhang and Wang (2008). In this literature, however, the mid-term information revelation structure is exogenous.

It is straightforward to show that given the first and second period performance, the principal's posterior belief on player  $i$ 's ability  $\eta_i$  follows a normal distribution,

$$\eta_i|y_{A1}, y_{B1}, y_{A2}, y_{B2} \sim N\left(\frac{(\sigma_{\epsilon i}^2 m_0 + \sigma_0^2 y_{i1}) + \sigma_0^2 (y_{i2} - \hat{a}_i)}{\sigma_{\epsilon i}^2 + 2\sigma_0^2}, \frac{\sigma_0^2 \sigma_{\epsilon i}^2}{\sigma_{\epsilon i}^2 + 2\sigma_0^2}\right), \quad (1)$$

where  $\hat{a}_i$  is the principal's expectation on agent  $i$ 's effort level.

Then, without loss of generality, player A would win the tournament (or promotion) if

$$(\eta_A - \eta_B)|y_{A1}, y_{B1}, y_{A2}, y_{B2} \geq 0. \quad (2)$$

Assume that the winner of the tournament receives the payoff  $M$ , and the loser receives the payoff  $m$ , where  $M > m > 0$ . If the expected ability is the same, the principal promotes each agent with equal probability, a half. Computing the optimal  $M$  and  $m$  is not the focus of this paper. Thus, I assume that  $M$  and  $m$  are exogenously given.

Note that  $\sigma_{\epsilon i}^2$  measures the uncertainty or risk in player  $i$ 's task. In the following analyses, I will consider two different cases: (i) where the principal determines the level of risk ( $\sigma_{\epsilon i}^2$ ) and (ii) where the agents choose the level of risk ( $\sigma_{\epsilon i}^2$ ).

The timing of the game is as follows. First, the principal (or the agents) chooses the level of risk ( $\sigma_{\epsilon i}^2$ ). Then, both the principal and the agents observe the first period performance. In the second period, the agents choose their effort simultaneously. After observing the second period performance of the agents, the principal updates her belief on the agents' ability ( $\eta_i$ ), and promotes the agent with higher expected ability.

### 3 Principal's Choice of Uncertainty: Sorting vs. Incentive

This section considers a case where the principal selects the level of risk. From the symmetry of the model, assume  $\sigma_{\epsilon A}^2 = \sigma_{\epsilon B}^2 = \sigma_{\epsilon}^2$ . The principal has two objectives: (i) inducing more effort by the agents and (ii) promoting the agent with higher ability. The first objective reflects the incentive effect of the promotion, and the second reflects the sorting effect of the promotion.

For precise sorting (that is, to reduce the type I error), the principal must minimize the variance of the expected posterior belief. From (1), it is straightforward to show that the variance

of the posterior belief on the agents' ability is increasing in  $\sigma_\epsilon^2$ . Therefore, for precise sorting, the principal should choose  $\sigma_\epsilon^2$  as small as possible.

If a smaller  $\sigma_\epsilon^2$  increases the agent's effort as well, then there would be no trade-off between the incentive effect and the sorting effect, and the principal will choose the minimum  $\sigma_\epsilon^2$ , that is, she will select as precise a performance measure as possible. However, if reducing  $\sigma_\epsilon^2$  decreases the agent's effort, there would be a potential trade-off between the incentive effect and the sorting effect, and the optimal  $\sigma_\epsilon^2$  can be strictly positive. Then, the principal may intentionally choose a noisy performance measure.

**Agents' Choice of Effort** To analyze the incentive effect of promotion, consider the agent's optimal choice of effort given  $\sigma_\epsilon^2$ . Since  $\sigma_{\epsilon A}^2 = \sigma_{\epsilon B}^2 = \sigma_\epsilon^2$ , from (1) and (2), without loss of generality, player  $A$  would win the tournament if

$$\begin{aligned} y_{A1} + (\eta_A + a_A + \epsilon_{A2} - \hat{a}_A) - y_{B1} - (\eta_B + a_B + \epsilon_{B2} - \hat{a}_B) &\geq 0 \\ &\Downarrow \\ (\eta_A + \epsilon_{A2} - \eta_B - \epsilon_{B2}) &\geq (\hat{a}_A - \hat{a}_B) - (a_A - a_B) - (y_{A1} - y_{B1}). \end{aligned}$$

Note that conditional on the first period performance  $y_{A1}$  and  $y_{B1}$ ,

$$(\eta_A + \epsilon_{A2} - \eta_B - \epsilon_{B2})|_{y_{A1}, y_{B1}} \sim N\left(\frac{\sigma_0^2(y_{A1} - y_{B1})}{\sigma_\epsilon^2 + \sigma_0^2}, \frac{2\sigma_\epsilon^4 + 4\sigma_0^2\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_0^2}\right). \quad (3)$$

Denote the cumulative distribution function of  $(\eta_A + \epsilon_{A2} - \eta_B - \epsilon_{B2})$  by  $F$ . Then, in the beginning of the second period, player A's probability of winning given the first period performance is

$$P_A = 1 - F((\hat{a}_A - \hat{a}_B) - (a_A - a_B) - (y_{A1} - y_{B1})).$$

Likewise, player B's probability of winning given the signals is

$$P_B = F((\hat{a}_A - \hat{a}_B) - (a_A - a_B) - (y_{A1} - y_{B1})).$$

Then, given the first period performance, player A's maximization problem in the second period is as follows:

$$\max_{a_A \geq 0} U_A = m + P_A(M - m) - g(a_A).$$

Denote the pdf of  $F$  by  $f$ . Then, the first order condition for the agent A's optimal effort  $a_A^*$  is

$$\begin{aligned}\frac{\partial U_A}{\partial a_A} &= f((\hat{a}_A - \hat{a}_B) - (a_A^* - a_B^*) - (y_{A1} - y_{B1}))(M - m) - g'(a_A^*) \\ &= f(-(y_{A1} - y_{B1}))(M - m) - g'(a_A^*) \leq 0,\end{aligned}\tag{4}$$

where equality holds if  $a_A^* > 0$ . The second equality follows from the rational expectations assumption, that is,  $\hat{a}_A = a_A^*$  and  $\hat{a}_B = a_B^*$ .

Likewise, the first order condition for the agent B's optimal effort  $a_B^*$  is

$$\frac{\partial U_B}{\partial a_B} = f(-(y_{A1} - y_{B1}))(M - m) - g'(a_B^*) \leq 0,\tag{5}$$

where equality holds if  $a_B^* > 0$ .

Then, the agents' optimal choice of effort can be characterized as follows:

**Proposition 1** (i) *Regardless of the first period performance  $(y_{A1}, y_{B1})$ , both players choose the same level of effort ( $a_A^* = a_B^* = a^*$ ).*

(ii) *If  $|y_{A1} - y_{B1}|$  increases, then the effort level decreases.*

**Proof.** See appendix. ■

The first result is due to the additiveness of ability and effort in the agent's performance, and simplifies our analysis greatly. The second result is the well-known asymmetric tournament effect. That is, if  $y_{A1} - y_{B1}$  increases, the posterior expected difference in the agents' ability increases. Therefore, an agent with lower expected ability would work less because he is likely to lose anyway. Also, the agent with higher expected ability works less because he is likely to win anyway.

**Principal's Optimal Choice of Uncertainty** Given (3) and (4), I can characterize the effect of  $\sigma_\epsilon^2$  on the agent's expected optimal effort  $E[a^*]$  as follows:

**Proposition 2** *There exists  $s(\sigma_0^2) > 0$  such that*

(i) *if  $\sigma_\epsilon^2 > s(\sigma_0^2)$ ,  $E[a^*]$  decreases in  $\sigma_\epsilon^2$ ,*

- (ii) if  $\sigma_\epsilon^2 < s(\sigma_0^2)$ ,  $E[a^*]$  increases in  $\sigma_\epsilon^2$ ,
- (iii) if  $\sigma_\epsilon^2 = 0$ , then  $E[a^*] = 0$ ,
- (iv)  $s(\sigma_0^2)$  increases in  $\sigma_0^2$ .

**Proof.** See appendix. ■

Therefore, when  $\sigma_\epsilon^2$  is large enough relative to  $\sigma_0^2$ , raising the uncertainty  $\sigma_\epsilon^2$  decreases the agents' effort. In this case, as discussed above, I do not have a trade-off between sorting and incentive. Therefore, the principal would not choose  $\sigma_\epsilon^2 > s(\sigma_0^2)$ . Intuitively, if  $\sigma_\epsilon^2$  is large enough, even a large difference in the second period performance does not imply a large difference in ability. That is, with large uncertainty in the performance measure, an agent's higher effort or higher performance cannot convince the principal that the agent has higher ability. Therefore, the agents lose the incentives for effort.

However, if  $\sigma_\epsilon^2$  is small enough relative to  $\sigma_0^2$ , raising the uncertainty  $\sigma_\epsilon^2$  increases the agents' effort. In this case, there exists a trade-off between sorting and incentive. Intuitively, if  $\sigma_\epsilon^2$  is small enough (that is, performance measure is precise), even a small difference in the first period performance implies a statistically significant difference in the agents' ability and reduces the agents' effort.

If  $\sigma_\epsilon^2 = 0$ , then the difference in the first period reveals the difference in ability for sure. Thus, there is no incentive for any effort in the second period. Therefore, assuming that the principal never wants the agents to choose zero effort, the principal would not choose  $\sigma_\epsilon^2 = 0$ . Thus, even without specifying the principal's objective function, the optimal  $\sigma_\epsilon^2$  for the principal must be finite and strictly positive. That is,

**Corollary 1** *The optimal  $\sigma_\epsilon^2$  for the principal is  $0 < \sigma_\epsilon^2 < s(\sigma_0^2)$ .*

Note that even when the principal has the choice of a precise performance measure with  $\sigma_\epsilon^2 = 0$ , the principal would still intentionally choose a noisy performance measure. This result is related to the recent literature on the mid-term performance revelation discussed in the beginning. For

example, Ederer (2009) shows that revealing mid-term performance can create an asymmetric tournament and reduce the agents' incentives. Proposition 2 and corollary 1 show that a similar intuition drives the principal to choose a noisy performance measure intentionally. This contrasts with the conventional wisdom in agency models that the more precise the performance measure is, the better. Moreover, unlike the previous studies that have focused on the incentive effect, proposition 2 shows that the sorting effect of promotion tournaments may directly conflict with the incentive effect.

## 4 Agent's Optimal Choice of Risk: Risk-Taking vs. Assurance

In this section, I allow the agents, not the principal, to choose their own risk,  $\sigma_{\epsilon_i}^2$ , in the beginning of the game. That is, given that the winner is decided by the principal's subjective belief about each agent's ability, I am interested in how much risk the agents are willing to take. Note that the choice of risk is public information, that is, observable to the principal and the other agent, and affects the principal's (posterior) belief.

Solving backwards, from (1), in the end of the second period, agent A would win the tournament if

$$\begin{aligned} \Delta_A &\equiv E[\eta_A - \eta_B | y_{A1}, y_{B1}, y_{A2}, y_{B2}] \\ &= \frac{\sigma_{\epsilon_A}^2 m_0 + \sigma_0^2 y_{A1} + \sigma_0^2 (y_{A2} - \hat{a}_A)}{\sigma_{\epsilon_A}^2 + 2\sigma_0^2} - \frac{\sigma_{\epsilon_B}^2 m_0 + \sigma_0^2 y_{B1} + \sigma_0^2 (y_{B2} - \hat{a}_B)}{\sigma_{\epsilon_B}^2 + 2\sigma_0^2} \geq 0. \end{aligned} \quad (6)$$

Let us denote the cdf of  $\Delta_A$  given the first period performance by  $H$ . Then, in the second period, given  $\sigma_{\epsilon_A}^2$  and  $\sigma_{\epsilon_B}^2$  and the first period performance  $y_{A1}$  and  $y_{B1}$ , agent A would choose his effort level to maximize the expected utility as follows:

$$\max_{a_A \geq 0} U_A = m + (1 - H(0))(M - m) - g(a_A). \quad (7)$$

Finally, the agents would choose the level of risk in their tasks  $\sigma_{\epsilon_A}^2$  and  $\sigma_{\epsilon_B}^2$  in the beginning of the game to maximize their expected utility. Even though a closed form solution for the optimal risk choice is not available, assuming that there exists a symmetric equilibrium, the following results arise.

**Proposition 3** *If there exists a symmetric equilibrium,  $\sigma_{\epsilon A}^2$  and  $\sigma_{\epsilon B}^2$  are finite. Also,  $E[a_A^*]$  and  $E[a_B^*]$  are strictly positive.*

**Proof.** See appendix. ■

Intuitively, in a tournament, each agent has an incentive to increase their risk  $\sigma_{\epsilon i}^2$  to have a higher performance than the other agent. However, when the winner is decided by the principal's belief, not by the realized performance, if  $\sigma_{\epsilon A}^2$  and  $\sigma_{\epsilon B}^2$  are infinite, higher performance does not increase the principal's belief about the agents' ability. Therefore, to win this tournament, each agent has an incentive to reduce the risk and work hard, and the optimal risk is finite.

Since many managers, especially fund managers, have increasingly large discretion on the choice of risk in their strategy, the question of whether promotion tournaments among them would lead to excessive risk-taking has become an important issue, especially after the recent financial crisis. Hvide (2002), for example, shows a negative result that in a standard tournament model, allowing the agents to choose the risk leads to infinite risk-taking and zero effort. In this case, note that the promotion tournament provides neither the incentive effect nor the sorting effect.

This paper restores some optimism. That is, as long as the principal (or firm) can observe the managers' risk-taking behavior and promote one based on subjective belief about the managers' ability, the managers do not take too excessive risk. Proposition 3, however, does not tell us whether the equilibrium risk level is socially optimal.

## 5 Conclusion

Promotions are directly tied to the hierarchical organization structure of a firm, and are a main source of wage growth for workers. Therefore, a promotion tournament must sort out the best worker for the higher-ranked job, and provide work incentives to the employees. With uncertainty about workers' ability, the sorting problem inevitably involves the supervisor's subjective belief about workers' ability and affects the promotion decision. However, the previous literature has

largely focused on sport-like tournaments where the agents' performance, not the principal's belief, determines the winner.

This paper provides a simple model of tournament to analyze the importance of a supervisor's subjective belief in promotion tournaments and the sorting effect of promotions. This paper shows that the principal would intentionally choose a noisy performance measure, and that the agents would not take infinite risk. These results suggest that a tournament based on the principal's subjective belief can be better than a tournament based on precise and objective performance measure in terms of both the sorting effect and the incentive for effort.

## References

- [1] Aoyagi, M., 2010, "Information feedback in a dynamic tournament," *Games and Economic Behavior*, 70, pp.242-260.
- [2] Biais, B. and C. Casamatta, 1999, "Optimal Leverage and Aggregate Investment," *Journal of Finance*, 54, pp.1291-1323.
- [3] Cabral, L., 2003, "R&D Competition When Firms Choose Variance", *Journal of Economics and Management Strategy*, 12, 139-150.
- [4] Cai, G., Wurman, P.R. and Chao, X., 2007, 'The non-existence of equilibrium in sequential auctions when bids are revealed', *Journal of Electronic Commerce Research*, vol. 8, pp. 141-156.
- [5] Chevalier, J. and G. Ellison, 1997, "Risk Taking by Mutual Funds as a Response to Incentives." *Journal of Political Economy*, 105, pp. 1167–1200.
- [6] Diamond, P.,1998, "Managerial Incentives: On the Near Linearity of Optimal Compensation," *Journal of Political Economy*, 106, pp.931-957.
- [7] Ederer, F., 2009. Feedback and motivation in dynamic tournaments. Working paper, UCLA.
- [8] Ehrenberg, R., and Boganno, M., 1990, "Do tournaments have incentive effects?" *Journal of Political Economy*, 98(6), 1307-1323.
- [9] Gershkov, A., Perry, M., 2006. Tournaments with midterm reviews. Working paper, University of Bonn and Hebrew University.
- [10] Goltsman, M., Mukherjee, A., 2006. Information disclosure in multistage tournaments. Working paper. University of Western Ontario and Bates White LLC.
- [11] Hellwig, M., 1994, "A Reconsideration of the Jensen-Meckling Model of Outside Finance." Discussion Paper no. 9422, WWZ University of Basel.

- [12] Holmstrom, Bengt., 1982, "Managerial Incentive Schemes-a Dynamic Perspective."In Essays in Economics and Management in Honour of Lars Wahlbeck.Helsinki: Svenska Handelshögskolan.
- [13] Hvide, H.K., 2002, "Tournament Rewards and Risk Taking," *Journal of Labor Economics*, 20(4), pp.877-898.
- [14] Lazear, E., and S. Rosen, 1981, "Rank Order Tournaments as Optimal Labor Contracts," *Journal of Political Economy*, 89(5), 841–864.
- [15] Mezzetti, C., Pekec, A. and Tsetlin, I., 2006, 'Sequential vs. single-round uniform-price auctions', *Games and Economic Behavior*, vol. 62, pp. 591-609.
- [16] Moldovanu, B. and Sela, A., 2006, 'Contests architecture', *Journal of Economic Theory*, vol. 126, pp. 70-96.
- [17] Nieken, P. and Sliwka, D., 2008, "Risk-Taking Tournaments: Theory and Experimental Evidence," IZA Discussion Papers 3400, Institute for the Study of Labor (IZA).
- [18] O'Keeffe, M., W.K. Viscusi, R.J. Zeckhauser, 1984, "Economic Contests: Comparative Reward Schemes," *Journal of Labor Economics*, 2(1), pp. 27-56.
- [19] Palomino, F., and A. Prat, 2003, "Risk taking and optimal contracts for money managers," *Rand Journal of Economics*, 34(1), pp. 113-137.
- [20] Yildirim, H., 2005. Contests with multiple rounds. *Games Economic Behavior*. 51, 213–227.
- [21] Wang, R., Zhang, J., 2009. The role of information revelation in elimination contests. *The Economic Journal*, 119, pp. 613-641.

## APPENDIX

**Proof of Proposition 1** (i) From (4) and (5), the first order conditions for players  $A$  and  $B$  are symmetric. Therefore,  $a_A^* = a_B^*$ .

(ii) Without loss of generality, consider the posterior belief about player  $A$ 's ability. The posterior mean and variance of  $\eta_A$  conditional on the first period performance are

$$m_{A1} = \frac{\frac{1}{\sigma_0^2}m_0 + \frac{1}{\sigma_\epsilon^2}y_{A1}}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_\epsilon^2}} = \frac{\sigma_\epsilon^2 m_0 + \sigma_0^2 y_{A1}}{\sigma_\epsilon^2 + \sigma_0^2} \quad (\text{A.1})$$

$$\sigma_1^2 = \left( \frac{1}{\sigma_0^2} + \frac{1}{\sigma_\epsilon^2} \right)^{-1} = \frac{\sigma_0^2 \sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_0^2}. \quad (\text{A.2})$$

The posterior mean and variance of  $\eta_A$  conditional on the first and the second period performance are

$$\begin{aligned} m_{A2} &= \frac{\frac{1}{\sigma_1^2}m_{A1} + \frac{1}{\sigma_{\epsilon A}^2}(y_{A2} - \hat{a}_A)}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_{\epsilon A}^2}} = \frac{\sigma_{\epsilon A}^2 m_{A1} + \sigma_1^2(y_{A2} - \hat{a}_A)}{\sigma_{\epsilon A}^2 + \sigma_1^2} \\ &= \frac{\sigma_{\epsilon A}^2 \frac{\sigma_\epsilon^2 m_0 + \sigma_0^2 y_{A1}}{\sigma_\epsilon^2 + \sigma_0^2} + \frac{\sigma_0^2 \sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_0^2}(y_{A2} - \hat{a}_A)}{\sigma_{\epsilon A}^2 + \frac{\sigma_0^2 \sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_0^2}} \\ &= \frac{\sigma_{\epsilon A}^2(\sigma_\epsilon^2 m_0 + \sigma_0^2 y_{A1}) + \sigma_0^2 \sigma_\epsilon^2 (y_{A2} - \hat{a}_A)}{\sigma_{\epsilon A}^2 (\sigma_\epsilon^2 + \sigma_0^2) + \sigma_0^2 \sigma_\epsilon^2}, \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} \sigma_{A2}^2 &= \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_{\epsilon A}^2} \right)^{-1} = \left( \frac{\sigma_{\epsilon A}^2 + \frac{\sigma_0^2 \sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_0^2}}{\frac{\sigma_0^2 \sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_0^2} \sigma_{\epsilon A}^2} \right)^{-1} \\ &= \frac{\sigma_0^2 \sigma_\epsilon^2 \sigma_{\epsilon A}^2}{\sigma_{\epsilon A}^2 (\sigma_\epsilon^2 + \sigma_0^2) + \sigma_0^2 \sigma_\epsilon^2}. \end{aligned} \quad (\text{A.4})$$

Without loss of generality, suppose that  $y_{A1} - y_{B1} > 0$ . From (A.1), the mean of  $(\eta_A + \epsilon_{A2} - \eta_B - \epsilon_{B2})$  conditional on the first period performance is  $\frac{\sigma_0^2(y_{A1} - y_{B1})}{\sigma_\epsilon^2 + \sigma_0^2}$ . Since  $(y_{A1} - y_{B1}) > \frac{\sigma_0^2(y_{A1} - y_{B1})}{\sigma_\epsilon^2 + \sigma_0^2} > 0 > -(y_{A1} - y_{B1})$ , from the symmetry of  $f$ , if  $(y_{A1} - y_{B1})$  increases,  $f(-(y_{A1} - y_{B1}))$  decreases. Therefore, the second period effort decreases with  $(y_{A1} - y_{B1})$ .

**Proof of Proposition 2** (i) and (ii) From (A.1) and (A.2),

$$f(-(y_{A1} - y_{B1})) = \frac{1}{\sqrt{2\pi \frac{2\sigma_\epsilon^4 + 4\sigma_0^2\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_0^2}}} \exp\left(-\frac{\left(- (y_{A1} - y_{B1}) - \frac{\sigma_0^2(y_{A1} - y_{B1})}{\sigma_\epsilon^2 + \sigma_0^2}\right)^2}{2\frac{2\sigma_\epsilon^4 + 4\sigma_0^2\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_0^2}}\right).$$

Then, since  $g(a) = e^a - 1$ , from (4) and (5), we can get the closed form solution for the agent's optimal choice of effort as follows:

$$a_A^* = a_B^* = \log(M - m) - \log\left(\sqrt{2\pi \frac{2\sigma_\epsilon^4 + 4\sigma_0^2\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_0^2}}\right) - \frac{\left(\frac{\sigma_\epsilon^2 + 2\sigma_0^2}{\sigma_\epsilon^2 + \sigma_0^2}\right)^2}{2\frac{2\sigma_\epsilon^4 + 4\sigma_0^2\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_0^2}}(y_{A1} - y_{B1})^2.$$

Recall that  $E[X^2] = Var(X) + (E[X])^2$ . Since  $(y_{A1} - y_{B1}) \sim N(0, 2(\sigma_0^2 + \sigma_\epsilon^2))$ ,

$$\begin{aligned} E[a_i^*] &= \log(M - m) - \log\left(\sqrt{2\pi \frac{2\sigma_\epsilon^4 + 4\sigma_0^2\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_0^2}}\right) - \frac{\left(\frac{\sigma_\epsilon^2 + 2\sigma_0^2}{\sigma_\epsilon^2 + \sigma_0^2}\right)^2}{2\frac{2\sigma_\epsilon^4 + 4\sigma_0^2\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_0^2}}2(\sigma_0^2 + \sigma_\epsilon^2) \\ &= \log(M - m) - \log\left(\sqrt{4\pi \frac{\sigma_\epsilon^2(\sigma_\epsilon^2 + 2\sigma_0^2)}{\sigma_\epsilon^2 + \sigma_0^2}}\right) - \frac{1}{2\sigma_\epsilon^2}(2\sigma_0^2 + \sigma_\epsilon^2). \end{aligned}$$

It is straightforward to check that  $E[a_i^*]$  is maximized at  $\sigma_\epsilon^2 = s(\sigma_0^2) \equiv \frac{4}{3} \frac{\sigma_0^4}{\sqrt[3]{\frac{44}{27}\sigma_0^{12} + 2\sigma_0^6}} + \sqrt[3]{\frac{44}{27}\sigma_0^{12} + 2\sigma_0^6}$ . Therefore, if  $\sigma_\epsilon^2 > s(\sigma_0^2)$ ,  $E[a^*]$  decreases in  $\sigma_\epsilon^2$ . And if  $\sigma_\epsilon^2 < s(\sigma_0^2)$ ,  $E[a^*]$  increases in  $\sigma_\epsilon^2$ .

(iii) Also, if  $\sigma_\epsilon^2 = 0$ , from (4) and (5),  $a^* = 0$  for all  $y_{A1}$  and  $y_{B1}$ . Therefore,  $E[a^*] = 0$ .

(iv) It is straightforward to check that  $s'(\sigma_0^2) > 0$  for all  $\sigma_0^2 > 0$ .

**Proof of Proposition 3** Note that in the beginning of the second period, before the second period performance is realized,  $\Delta_A$  is a random variable. In particular, since  $y_{i2} = \eta_i + a_i + \epsilon_{i2}$ ,

$\Delta_A$  follows a normal distribution with

$$E[\Delta_A|y_{A1}, y_{B1}] = \frac{\sigma_{\epsilon_A}^2 m_0 + \sigma_0^2 y_{A1} + \sigma_0^2 \left( \frac{\sigma_{\epsilon_A}^2 m_0 + \sigma_0^2 y_{A1}}{\sigma_{\epsilon_A}^2 + \sigma_0^2} + a_A - \widehat{a}_A \right)}{\sigma_{\epsilon_A}^2 + 2\sigma_0^2} - \frac{\sigma_{\epsilon_B}^2 m_0 + \sigma_0^2 y_{B1} + \sigma_0^2 \left( \frac{\sigma_{\epsilon_B}^2 m_0 + \sigma_0^2 y_{B1}}{\sigma_{\epsilon_B}^2 + \sigma_0^2} + a_B - \widehat{a}_B \right)}{\sigma_{\epsilon_B}^2 + 2\sigma_0^2}, \quad (\text{A.5})$$

$$\begin{aligned} \text{Var}[\Delta_A|y_{A1}, y_{B1}] &= \left( \frac{\sigma_0^2}{\sigma_{\epsilon_A}^2 + 2\sigma_0^2} \right)^2 \left( \frac{\sigma_0^2 \sigma_{\epsilon_A}^2}{\sigma_{\epsilon_A}^2 + 2\sigma_0^2} + \sigma_{\epsilon_A}^2 \right) \\ &\quad + \left( \frac{\sigma_0^2}{\sigma_{\epsilon_B}^2 + 2\sigma_0^2} \right)^2 \left( \frac{\sigma_0^2 \sigma_{\epsilon_B}^2}{\sigma_{\epsilon_B}^2 + 2\sigma_0^2} + \sigma_{\epsilon_B}^2 \right). \end{aligned} \quad (\text{A.6})$$

Let us denote the pdf of  $H$  by  $h$ . Then, from (7), given the first period performance  $y_{A1}$  and  $y_{B1}$  and the choice of risk  $\sigma_{\epsilon_A}^2$  and  $\sigma_{\epsilon_B}^2$ , the first order condition for  $a_A^*$  is

$$\frac{\partial U_A}{\partial a_A} = - \int_{-\infty}^0 h(s) \frac{(s - E[\Delta_A|y_{A1}, y_{B1}])}{\text{Var}[\Delta_A|y_{A1}, y_{B1}]} \frac{\partial E[\Delta_A|y_{A1}, y_{B1}]}{\partial a_A} ds (M - m) - g'(a_A^*) \leq 0, \quad (\text{A.7})$$

where equality holds if  $a_A^* > 0$ .

Suppose that  $\sigma_{\epsilon_A}^2 = \sigma_{\epsilon_B}^2 = \infty$ . Then, from (6),  $\Delta_A = 0$  for all  $y_{A1}, y_{B1}, y_{A2}$ , and  $y_{B2}$ . Therefore, each agent will get promoted randomly with probability a half. That is,  $H(0) = \frac{1}{2}$  in (7). Thus, from rational expectation, we have  $a_A^* = \widehat{a}_A = 0$ . Similarly,  $a_B^* = \widehat{a}_B = 0$ .

Assuming there exists a symmetric equilibrium, it is sufficient to show that agent  $A$  can profitably deviate to a finite  $\sigma_{\epsilon_A}^2$ . Given  $\sigma_{\epsilon_B}^2 = \infty$  and  $a_B^* = \widehat{a}_A = \widehat{a}_B = 0$ , from (A.5),

$$E[\Delta_A|y_{A1}, y_{B1}] = \frac{\sigma_{\epsilon_A}^2 m_0 + \sigma_0^2 y_{A1} + \sigma_0^2 \left( \frac{\sigma_{\epsilon_A}^2 m_0 + \sigma_0^2 y_{A1}}{\sigma_{\epsilon_A}^2 + \sigma_0^2} + a_A^* \right)}{\sigma_{\epsilon_A}^2 + 2\sigma_0^2} - m_0. \text{ Then, in the beginning of the first period,}$$

before the first period performance is realized,  $E[\Delta_A] = \frac{\sigma_{\epsilon_A}^2 m_0 + \sigma_0^2 m_0 + \sigma_0^2 \left( \frac{\sigma_{\epsilon_A}^2 m_0 + \sigma_0^2 m_0}{\sigma_{\epsilon_A}^2 + \sigma_0^2} + E[a_A^*] \right)}{\sigma_{\epsilon_A}^2 + 2\sigma_0^2} - m_0 = \sigma_0^2 \frac{E[a_A^*]}{2\sigma_0^2 + \sigma_{A\epsilon}^2}$ . Therefore, assuming  $E[a_A^*] > 0$ , agent  $A$  can increase the probability of winning to more than a half by choosing a finite  $\sigma_{A\epsilon}^2$ . Finally, given  $\sigma_{A\epsilon}^2$  is finite,  $E[a_A^*]$  must be strictly positive from (A.7) since  $a_A^*$  is bounded below by zero and strictly positive if  $y_{A1}$  is large enough.