1 Introduction

In many cases, a competition is comprised of several linked subcontests, combinations of which exhibit complementarity, such that the benefits accrue only by having won multiple subcontests in combination. This will cause differences in valuation based on what complementary subsets have already been obtained, or are thought likely to be obtained in the future. This can extend to cases where different people have different complimentary sets.

An example of such a situation would be a battle between two different interest groups, developers, who wish to connect two towns, and environmentalists, who wish to maintain a forest corridor for wildlife. Each side spends money trying to influence the zoning board in each area, however, crucially, the winner is not strictly the one who spends more money in an area. Instead, each side has a chance of winning each area proportional to the amount they invest in an area compared with the total spending on the area. This assumption is reasonable for this situation, as efforts to influence a zoning board will be imperfect, with the argument likely to sway a member being unknown to either interest group ahead of time. Additionally, random outside factors will likely play a large role in determining a winner. This allows for a pure strategy equilibrium to exist, which we will compute. Other cases where this structure may be useful would be a military campaign, where one side is attempting to build a line through enemy territory. In such a case, great allocation of troops does not insure success; however, the probability of victory does increase with having a greater ratio of troops in a battle. Other possible cases would be drug interdiction patrols, being a contest between police and drug smugglers, and computer network security, where data can be rerouted away from a compromised server so long as an intact series of links exists.

Previous work has been done on similar games. The most similar example would be the Colonel Blotto game. [6] In this game, several hills are being contested by two military forces, who are attempting to seize control. Each hill has a value, each side has a fixed number of soldiers, and the side with greater forces automatically wins the battle at each hill, with the objective to achieve the greatest total value of captured hills. Though this differs from the game under consideration, it is the starting point for this style of game. Colonel Blotto, as the winner of each hill is deterministic, results in mixed strategies being the optimal solution in most cases, as opposed to the single optimal strategy the we obtain from the continuous probability distribution of victory for each area. A good general version of this is solved by Roberson [7]. Such games are useful in other contexts with competition involving finite resources, such as lobbying and research and development between corporate labs [8]. Colonel Blotto also has a large number of variants, for example a stochastic version with victory probability being a function of troop commitments. Such games have been studied both theoretically and experimentally, with most beginning players quickly moving toward near-optimal play [9]. A survey of recent results is given by Kovenock and Roberson [2].

Two other recent relevant papers are one by Hagedorn and another by Eso and Szentes. In Hagedorn's paper [11], he is looking at the benefits to an auctioneer of knowing the bidders' valuations of the object offered. In this case the auctioneer knows both the individual valuations for each player, but the players do not know each other's valuation. This allows the auctioneer to manipulate the auction to maximize bids, increasing his own benefit. Interestingly, it turns out that offering differing information to the two bidders is always more beneficial for the auctioneer compared to offering both the average of these two information sets. Thus, the auctioneer benefits by deliberately creating unbalanced competitions. Similarly, we will see that varying the structure of the auction in our paper will vary the gains to the auctioneer. However, as there is not an information set that differs between players, but rather the structure of the game itself, any imbalances will be known to both contestants as well as the auctioneer.

Similarly, Eso and Szentes consider handicaps for auctions to balance out informational asymmetries [12]. This is a somewhat more complicated auction structure; however, it is another example of how an auctioneer can manipulate the auction process for personal gain. In this case there are many bidders, all of whom have incomplete information on other player's valuations, and the auctioneer has some ability to release signals on the values to individual bidders place on the object. The question then becomes how much of these signals, if any, should the auctioneer release, and what fees should he charge for this information. None of these games exhibit the complementarity that is of particular interest in this case; however, they provide a useful baseline comparison. Our game is related in that though the winners of previous rounds are known to both players, the information may be more useful to one player depending on the structure of the ordering of the auctions.

The paper is laid out as follows. First, an example is described, showing the general method of play. The model is then solved, with specific illustrative examples detailed, with the players and strategies explained. A discussion then follows of how to solve cases in general. Results are presented, followed by a short discussion of when this game is a more appropriate model compared to Colonel Blotto games. We then extend the game to include decision making in choosing the auction structure, with multiple auctioneers permitted.

2 A Brief History of Hex

In the canonical form, Hex is played by two players on an 11 by 11 space grid of hexagonal cells, the two players conventionally labeled Black and White. Each player alternates claiming an unclaimed cell of the board. The objective is for Black to connect the two black sides of the board with a path of his pieces, while White attempts to connect the two white sides of the board with his pieces. This image shows a game in progress [1].

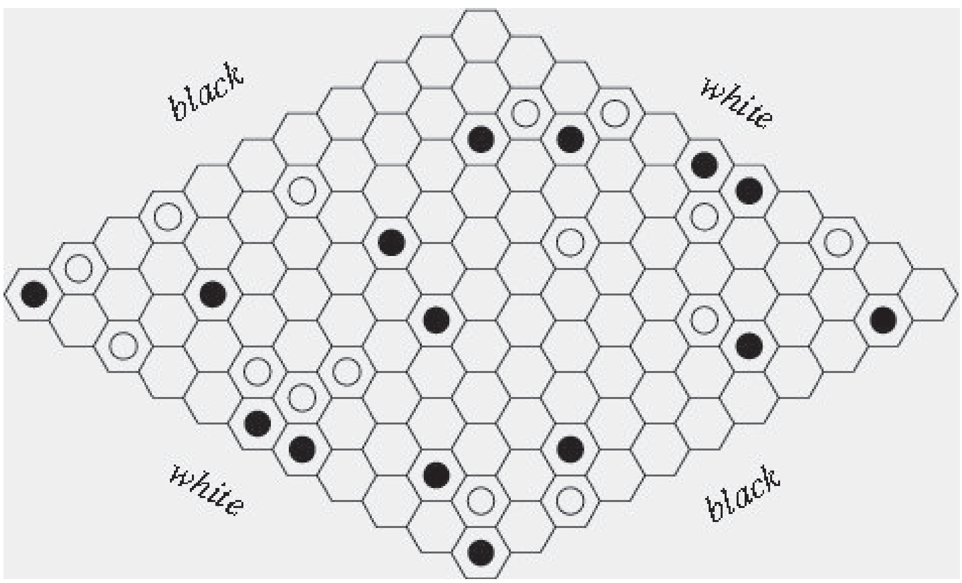

Figure 1: Game of Hex in Progress

As this game continues, eventually a state is reached where one player has won by completing a path connecting the two edges the player has been assigned. Draws are impossible, as if it has become impossible for one player to make a connection, all of their routes have been cut off, which means there must be a continuous path from the opposing player.

This game occupies an unusual place in the history of game theory for several reasons. Firstly, one of the two independent inventors of the game in the 1940's was John Nash, along with Piet Hein. Secondly, this game was proven to have a perfect strategy for the first player well before such a strategy was found, even for smaller cases, with the 11 by 11 case still unsolved. The argument proceeds from contradiction, as owning a space is always beneficial, if the second player had a winning strategy, the first player could copy this winning strategy with the advantage of already having a space. As there are no ties possible, this results in the first player having a guaranteed victory [3]. This strategy stealing argument is a common one in combinatorial game theory, giving proofs of either a player, usually the first, having some certain victory, or that both players are able to force a draw.

Furthermore, the complexity of this game has proven very difficult, with determining a perfect strategy being known to be harder than NP-Complete [4]. Thus, though a perfect strategy may be able to be checked in polynomial time, finding such a solution is almost certainly impossible to do in a length of time that is a polynomial function of the grid size. This game is still studied, including new playing algorithms that do not attempt to solve the game using combinatorial game theory. [5]

3 Hex And Complementarity

The structure of Hex provides a useful starting point for building complimentarities. As each player is attempting to connect different sides, winning combinations will vary, and the value of an area will depend on the use of this area in creating a winning path. However, the canonical version of Hex is computationally difficult. In order to have a solvable problem, we will consider a small case. In this case, there are 4 areas being contested, forming a small grid. As noted, each side has two opposite edges that they wish to connect.

Our game differs from canonical Hex in that both players will play simultaneously. This is done by each player committing a resource to the areas currently being contested. After both players have committed resources, a winner for each cell is determined probabilistically, with each of the two players having a probability of winning a cell proportional to the amount of resources committed by that player. After a set of cells has been contested, a check is made to see if either player has won, receiving a prize V. If not, some new subset is contested, repeating this process until a winner of the overall contest is found. These subsets can range from containing single cells to containing all remaining cells.

As we see from this diagram, North and South should be equally valuable, as should East and West. However, North and South should be more

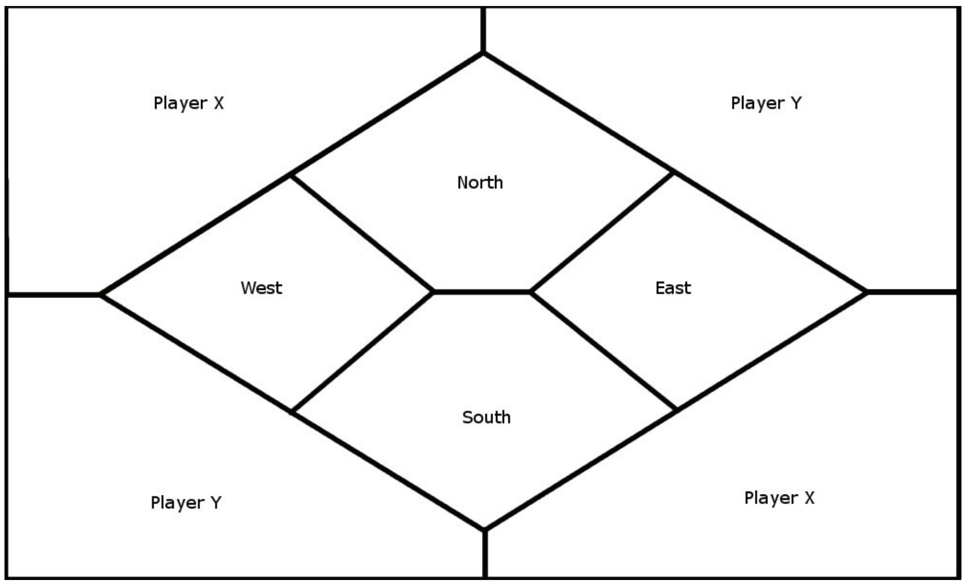

Figure 2: The Areas Being Contested

valuable than East and West, as North and South appear in more winning sets for each player than East and West do, and at least one of North and South is required, while victory with neither East nor West is possible. These asymmetries will provide the interesting distinctions in this paper.

4 An Illustrative Example

As an example, let the prize for completing a connection be 100, with all 4 areas being competed for simultaneously. Player X invests 20 in both the North and East areas, and only 1 in the West and South, trying to have overwhelming force in 2 areas which will provide victory. Meanwhile, Player Y invests 10 in both the North and South, and 5 in the East and West, realizing North and South have more winning combination available.

We can thus calculate the probabilities of X winning each area. In the North, he has 20 out of a total of 30 invested, for a $\frac{2}{3}$ chance of victory. In the East, he has 20 out of 25, or $\frac{4}{5}$. In the West he has 1 out of 6, or $\frac{1}{6}$, and in the South, he has 1 out of 11 total, for a probability of victory of $\frac{1}{11}$. Thus player X has a chance of victory of $\frac{2}{3}\frac{4}{5}+\frac{2}{3}\frac{1}{11}\frac{1}{5}+\frac{1}{6}\frac{1}{11}\frac{1}{3}$, the sum of his probabilities of winning North and East, North and South but not East, and West and South but not North. This covers all winning sets for player

X without double counting any sets, as winning North and East without regard to South and West covers winning North and East; North, East, and West; North, South, and East; as well as winning all four areas. Similarly, North and South but not East covers North and South and North, South, and West, with West and South but not North covers West and South and West, South, and East.

This gives a total chance of victory for player X of approximately .5505, so on average he will win 55.05 at a cost of 42, for a net gain of 13.05. Player Y will thus have .4495 probability of winning, and average winnings of 44.95. However, player Y only spent 30, for a net gain of 14.95.

5 The Model

We will build a model of this small case. This will be small enough to allow for finding explicit solutions.

The Players

A few assumptions are made about the players of the game. Let the set of players be $P = \{X, Y\}$. These players do not have a budget constraint. We will assume that the payout for victory, V, is identical for each player. This is done purely for simplifying purposes. We also assume that the players are risk neutral.

The Strategies

Before the contest begins, the auction structure S is announced. This is an ordered set of sets, labeled S_1, S_2, S_3, S_4, with each S_r being a set of areas to be contested in the rth round of the competition. The areas $\{N, S, E, W\}$ are partitioned into these 4 rounds. Rounds may be the null set or contain multiple areas. We will refer to the cardinality $|S_r|$ as s_r.

In each round r, each player will chose a vector of bids of length s_r with the bid for area i by player T being labeled T_i. We will require $T_i \geq 0$. [1]

We will denote the amount that player X invests in a given area as X_i, with the i being the area contested. Similarly, Y_i is the amount invested by player Y, and $Z_i = X_i + Y_i$. Thus if S_r is the set $\{N, S\}$, player X's bids are $\{1, 2\}$ and player Y's bids are $\{1, 1\}$, $X_N = 1, X_S = 2, Y_N = 1, Y_S = 1, Z_N = 2, Z_S = 3$. The winner of each area will be determined by a Tullock

[1] $T_i = 0$ will be limited to cases in which area i has been made irrelevant by earlier rounds.

contest function, so that player X will win area i with probability $\frac{X_i}{Z_i}$.

The Payoffs Each player will expend the resources invested into each area, with the player who completes a winning set gaining a benefit equal to V. Thus, the useful function to consider is the expected utility function, consisting of the probability of victory multiplied by the value V, minus the amount spent.

We will require notation for our calculations. Let $A = \{N, S, E, W\}$, the set of areas. Let $X^\star$ be the set of winning sets for player X, specifically

$$X^\star = \{\{N, S, E, W\}, \{N, S, E\}, \{N, S, W\},$$
$$\{N, E, W\}, \{S, E, W\}, \{N, E\}, \{N, S\}, \{S, W\}\}$$

Thus the probability of player X winning the prize is

$$\sum_{\alpha \in X^\star} \prod_{i \in \alpha, j \in A - \alpha} \frac{X_i}{Z_i} \frac{Y_j}{Z_j} \tag{1}$$

This makes player X's expected payoff the probability of victory multiplied by the prize, minus the total spent on all 4 areas

$$U_X \left(\{X_{S_r}, Y_{S_r}\}_{r=1\ldots4} \right) = \sum_{\alpha \in X^\star} \prod_{i \in \alpha, j \in A \setminus \alpha} \frac{X_i}{Z_i} \frac{Y_j}{Z_j} V - \sum_{i \in A} X_i \tag{2}$$

The utility function for player X is function of the round structure and the amounts invested in each area. Thus we take the amounts invested by each player in the first round, the amounts invested in the second round by each player, and so forth until all areas have been accounted for.

Player Y's winning set, $Y^\star$ is given by

$$Y^\star = \{\{N, S, E, W\}, \{N, S, E\}, \{N, S, W\},$$
$$\{N, E, W\}, \{S, E, W\}, \{N, W\}, \{N, S\}, \{S, E\}\}$$

Therefore

$$U_Y \left(\{X_{S_r}, Y_{S_r}\}_{r=1\ldots4} \right) = \sum_{\alpha \in Y^\star} \prod_{i \in \alpha, j \in A \setminus \alpha} \frac{Y_i}{Z_i} \frac{X_j}{Z_j} V - \sum_{i \in A} Y_i \tag{3}$$

6 Solving the Single Auctioneer Game

The possible structures may be divided into cases. We will refer to the structure of the number of areas to be auctioned in a round as being a main case, with the distribution of areas between these rounds as subcases. Subcases will be referred to as distinct if they do not replicate a distinct subcase up to symmetry. There are thus 8 possible main cases, with a total of 29 distinct subcases. We will analyze 2 of the main cases, those of all areas contested simultaneously and the 4 areas being contested sequentially.

Case 1: All four areas simultaneously

To solve this game, we will maximize the expected payoff for each player. The expected payoff is given by the sum of the probabilities of the 8 winning combinations for a player multiplied by the value of winning V, minus the expenditures made on the four contested areas. To do so, we obtain first order conditions and solve, obtaining the following result:

Proposition 1. *If all four areas are auctioned simultaneously, both players will employ a symmetric strategy of spending* $\frac{1}{8}$ *on each of the North and South areas, and spending* $\frac{1}{16}$ *on each of the East and West areas, giving each player a probability of victory of* $\frac{1}{2}$*.*

Proof : See Appendix (Case 1)

Case 2: Four areas sequentially

In order to solve for optimal bidding, we must work backwards, and obtain expected payout for all possible scenarios, which are then used in the expected payoff equations in the previous round. The order of the areas will influence the expected payoff to the players. Furthermore, as the two players have differing winning sets of areas, the ordering of these areas can bias the game in favor of one player.

In the cases where either North or South are the first area contested, the game will still be symmetric between players X and Y, however, the expected payoff for the players will be greater than in the case of all four areas being auctioned simultaneously.

Proposition 2. *All sequential structures consisting of auctioning one area at a time have greater average expected underdissipation than the simultaneous case.*

Proof: See Appendix (Case 2)

Dissipation is the distribution of investment by players in a contest. Both over- and under- dissipation may occur. Overdissipation consists of the amount expended by the players being greater than the expected payoff. The possibility of overdissipation in a game with Tullock contest functions

Type	Order	$E[U_X]$	$E[U_Y]$	EV[A]
4	NESW	.125	.125	.75
1-1-1-1	N-(E,W,S), S-(E,W,N)	.1797	.1797	.6719
1-1-1-1	E-W-(N,S), W-E-(N,S)	.1406	.1406	.7188
1-1-1-1	E-N-(W,S), W-S-(E,N)	.0731	.2315	.6954
1-1-1-1	W-N-(E,S), E-S-(W,N)	.2315	.0731	.6954

Table 1: Calculated Expected Values For Each Player and Auctioneer Under Simultaneous and Sequential Structures

has been explored in [10]. Underdissipation is the amount expended by players being less than the expected payoff.

Intuitively, this is due to the players having more information in the sequential case. Thus, the players can make more informed decisions, and not spend resources on areas which have become less valuable. This intuition also explains the following result, as due to the asymmetric nature of the winning sets, some information is more valuable to one player. Thus, if the information available will be more useful to one player, they will benefit.

Proposition 3. *In the sequential cases, asymmetric expected payoffs between players X and Y requires that one player has a winning set possible in an earlier round than the opponent. The first player with a possible winning set has a lower expected payoff.*

Proof : See Appendix (Case 2)

If one player has a winning set available earlier than the other player, this winning set must be won with less information available to the player, causing the asymmetry.

As an example, if the areas are to be auctioned sequentially, in the order East, North, West, South, player X has an expected payoff of approximately .0731, while player Y has an expectation of approximately .2315. This expected payoff for X is the lowest value that exists for X.

Note that is a necessary condition, not a sufficient one. For example, the order North, West, South, East does not result in asymmetric expected payouts.

7 More on Sequential Selling

In addition to the two extreme main cases discussed here, there are several more auction structures, those consisting of two or three rounds of competition. The methodology for solving these is very much similar to solving the four round cases: taking expected payoffs for the last round, using these to create expected payoff functions for the penultimate round, and solving the maximization of this to get the new expected payoffs, eventually obtaining the a priori expected payoff for each player. In the two and three round cases, we must also note how many areas are contested in each round. There are three main cases for two round structures, according to if there are one, two, or three areas in the first round; and three main cases for three rounds, depending on which round contains two areas. Each of these main cases has multiple subcases depending on the distribution of areas. Thus, in total we have 8 main cases and 29 subcases. When we solve these cases, we note a few trends that occur.

Remark 1
We see that in addition to the one round case, there are several other cases where each player has an expected payoff of .125; however, there is no case where the total expected payoff for the two players is less than .25. Thus, given a choice between the different structures, the best the auctioneer can expect to receive is .75. There are cases with individuals having lower expected payoffs, though the lowest individual expected payoff cases are in 4 round sequential cases that we have already seen.

Unfortunately, there does not seem to be a simple relationship between the different types of structure and the expected payoffs to the players. For example, a two round structure consisting of two areas in each round can have the lowest total expected payoff for the players of .25, however it also can have one of the highest at .5189, depending on the distribution of areas. In general, asymmetric structures appear to have greater total payoffs, however the relationship is fairly weak.

Remark 2
We can also generalize Proposition 3 to other auction structures. When the auction structure has asymmetric payoffs for the players, there is an asymmetry in the order of winning sets availability. This asymmetry can be having part of the set available earlier, or having more winning sets available after a number of rounds, in addition to having the only winning set available after a given round. The asymmetry can run in either direction in the general case, however, with the player with the first winning set sometimes having a lower expected payoff. This is due to some cases where one player has their first winning set occur in a single round, while the other player has

a winning set spread across multiple rounds.

Both of these results result directly from the calculations of expected payoffs. We can also use these calculations as a step towards a more complicated problem. If we wish the auction structure to be determined by a game played by the original owners of the areas of land, we will require the results obtained thus far to have payoff amounts for this overarching game. This will require using the expected amount spent on each area, rather than the amount spent by each player, however tis information is easily recovered from the calculations.

8 Multiple Auctioneers

We now also introduce the idea of multiple auctioneers. Each auctioneer owns some subset of the areas that are to be contested, and places them up for auction. Once the auction structure becomes known, the players then choose strategies for the first round, and the game proceeds as in the previous sections of this paper. Thus, we have added an initial round, in which each auctioneer decides on the structure of the auctioning of his areas.

For simplicity, we will consider a two round, two auctioneer structure, where the auctioneers can put each area up for auction in the early round, or hold onto the land until the later round. Decisions about which round to sell land in are made privately by each auctioneer. In order to find solutions, we must first find the expected amount spent on each area. To do so, we use the previously solved two round structures. In computing the expected values for each player, we found the expected expenditure by each player on each area. We now add these expected expenditures to find the total expected spending on each area for each auction structure.

These calculations assume that there is no discounting factor for the future. The two rounds of auctions can be taken to be close enough together that inflation and time inconsistent preferences can be reasonably ignored, while massive discounting would occur in future rounds, thus eliminating them from consideration. Thus we have 15 distinct possible structures, these being the partitions of the 4 areas into 2 rounds, as all 4 areas being in the first round will be identical to all 4 being in the second round. As we see from the table, the expected values for each area depend on the overall auction structure, without a clear trend towards an area being more or less valuable by being auctioned in the first round.

This is due to the complementarity of the areas for the players. Areas

Order	EV[N]	EV[S]	EV[E]	EV[W]
NESW	.25	.25	.125	.125
NSW-E	.24	.24	.08	.18
NSE-W	.24	.24	.18	.08
NEW-S	.375	.125	.09375	.09375
EWS-N	.125	.375	.09375	.09375
WS-NE	.296703	.093006	.104167	.10687
NE-WS	.093006	.296703	.10687	.104167
NW-SE	.093006	.296703	.104167	.10687
SE-NW	.296703	.093006	.10687	.104167
NS-EW	.25	.25	.125	.125
EW-NS	.25	.25	.125	.125
N-SEW	.25	.25	.125	.125
S-NEW	.25	.25	.125	.125
E-NWS	.145833	.145833	.125	.125
W-NES	.145833	.145833	.125	.125

Table 2: Expected Value of Each Area Under Two Round Structures

become more valuable for the auctioneers the more heavily contested they are by the players, and this depends on the players' desire to form a winning set. If an area is less likely to be important by being moved to the second round, such as when the area is likely to be irrelevant, it will decrease in expected value. Conversely, if an area is likely to still be important in the second round, it will become more valuable as the players will be contesting it more intensely.

We now begin breaking into cases, starting with the symmetric case. Throughout, we will mark each pure Nash Equilibria with an asterisk in the matrices.

8.1 Equal Area Distribution

First, we will consider cases with two areas being owned by each auctioneer.

Case MA1: NE,SW; SW,NE; NW,SE; SE,NW

First we will consider the case of 2 auctioneers, one owning the North and East areas, and the other controlling the South and West areas. For each of the 15 distinct possible auction structures, we calculate the expected amount spent on the areas owned by the first auctioneer (NE), and the second auctioneer, as shown in the table.

–	SW-	S-W	W-S	-SW
NE-	.375, .375	.32, .42	.21875, .46875	.40087, .19988
N-E	.42, .32	.375, .375*	.40357, .19717	.375, .375
E-N	.46875, .21875	.19717, .40357	.375, .375	.27083, .27083
-NE	.19988, .40087	.375, .375	.27083, .27083	.375, .375

Table 3: Case MA1

–	EW-	E-W	W-E	-EW
NS-	.5, .25*	.5, .1875	.5, .1875	.5, .25*
N-S	.48, .26	.389109, .211037	.389109, .211037	.5, .25*
S-N	.48, .26	.389109, .211037	.389109, .211037	.5, .25*
-NS	.5, .25*	.291667, .25	.291667, .25	.5, .25*

Table 4: Case MA2

Thus, the choices of the first auctioneer, in order, are to sell both the North and East in the first round, sell North in the first and East in the second round, sell East in the first and North in the second round, or sell both in the second round. Combined with the choice of the second auctioneer, we then take the expected spending on each area is from the table above, and sum the amounts for the North and East to find the first auctioneer's expected payoff. Summing the amounts for South and West will similarly give us the second auctioneer's expected payoff.

Looking at the table, we see that there is only one Nash equilibrium, N-E and S-W, giving both players an expected value of .375.

Case MA2: NS,EW

The other possibility for two auctioneers each owning two areas is for one to own the North and South, with the other having the East and West. In this case, the payout matrix is as follows

In addition to the NS-,EW-; NS-,-EW; -NS,EW-; N-S,-EW; S-N,-EW and -NS,-EW pure Nash equilibria, there are also an infinite number of mixed strategy equilibria consisting of assigning probability p to NS-, $1-p$ to -NS, q to EW-, and $1-q$ to -EW, as well as the second auctioneer choosing -EW, with the first randomly choosing any available strategy. All of these have expected payoff of .5 for the first auctioneer and .25 for the second.

8.2 Unequal Area Distribution

In these cases, one auctioneer owns three areas, while the other holds a single area.

Case MA3: EWS,N; NEW,S

–	N-	-N
EWS-	.5, .25	.5625, .125
EW-S	.5625, .125	.5, .25
ES-W, WS-E	.5, .24	.3040403, .296703
E-WS, W-ES	.50774, .093006	.395833, .145833
S-EW	.5, .25	.5, .25
-EWS	.5, .25	.5, .25

Table 5: Case MA3

We can see from the table that ES-W, WS-E, E-WS, and W-ES are all at strictly dominated by EW-S, and thus can be eliminated. Similarly, S-EW and -EWS are weakly dominated by both EWS- and EW-S, and strictly dominated by any (non-degenerate) mixed strategy between EWS- and EW-S, and thus can be eliminated. The remaining choices behave as Matching Pennies, and thus both auctioneers will employ a mixed strategy, the first auctioneer choosing EWS- and EW-S both with .5 probability, and the second auctioneer choosing between N- and -N, again with both having .5 probability. This results in the first auctioneer having expected value .53125 and the second .1875

Case MA4: NSW,E; NES,W

–	E-	-E
NWS-	.625, .125*	.66, .08
NS-W	.56, .18	.625, .125
NW-S, SW-N	.59375, .09375	.496579, .104167
N-WS, S-NW	.493876, .10687	.625, .125
W-NS	.625, .125*	.416666, .125
-NWS	.416666, .125	.625, .125

Table 6: Case MA4

There are the two pure equilibria NWS-,E- and W-NS,E-, along with mixed strategies consisting of the first auctioneer randomly choosing between the two. These all have expected payoffs of .625 for the first auctioneer, with the second receiving .125.

Proposition 4. *All equilibria result in the East and West areas being auctioned simultaneously*

PROOF As we see from the solutions, the equilibria consist of NSEW-, NS-EW, NEW-S, SEW-N, and EW-NS in various combinations.

If the East and West are auctioned simultaneously, this eliminates asymmetry between the two players. Thus we see coordinated behavior in forming an unbiased game by the two auctioneers.

Proposition 5. *Pure strategy equilibria yield the greatest total payoff to the auctioneers.*

PROOF The auction structures we obtain are precisely those one or two round structures that yielded the maximum expected payoff for the auctioneer in the single auctioneer case.

Corollary 6. *In a pure strategy equilibrium, owning the North or South are worth .25 each, the East and West being worth .125 each.*

PROOF The auction structures we obtain all have the expected values indicated.

9 Conclusions

For the players, we see that complementarity plays a major role in expected payoffs, both in the requirements of victory and in that this complementarity also occurs in the auction rounds. Differences in the order of possibly obtaining complementary sets create differences in the expected payoffs for the players.

Conversely, we see that the primary factor in the expected payoff for an auctioneer is owning the more valuable areas, North and South, and owning a larger number of areas. Little in the way of complementarity is seen in owning particular combinations, despite the players buying the areas having strong complementarity.

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