Rent-seeking group contests with one-sided private information

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Abstract

We consider a rent-seeking contest where players compete in groups for a prize of given value. One group has private information about its number of members, which can be either small or large. The other groups have possibly different but publicly known sizes. We show that the contest has a unique equilibrium in which players of the same group exert the same effort level. We also present the necessary and sufficient conditions under which each group exerts a positive or zero equilibrium effort.

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1 Introduction

Tullock (1980) introduced the rent-seeking contest in which players compete for a single prize by exerting efforts. The probability that a player wins the prize depends on his effort and that of the other players. A vast amount of papers have studied this model and extensions of it. For surveys, see Nitzan (1994), Lockard and Tullock (2001), Congleton *et al.* (2008) and Konrad (2009). In particular, Nitzan (1991) presented a model for a group contest, where players are members of groups of possibly different sizes, and the prize is awarded to one of these groups. The probability that a group wins the prize depends on the aggregate effort of this group and the aggregate efforts of the other groups. The prize has private good characteristics. If a group obtains the prize, a given sharing-rule is applied to divide it among its members, i.e. either equally, or proportionally to each member's effort, or based on a combination of this. Examples of group contests are situations where interest groups, like parties, localities, departments or industries, compete for a budget at the discretion of policy makers, or situations where research and development joint ventures of firms are racing to develop a new product that will create an economic rent.

This paper considers a group contest as introduced by Nitzan (1991). However, we suppose that one group has private information about its number of members, which can be either small (a small-type group) or large (a large-type group). The other groups have possibly different but publicly known sizes. All groups have a given identical valuation of the prize. For analytical simplicity, and to find unambiguous results, we suppose that the prize is equally shared among the members of the winning group. We show that our model has a unique pure-strategy Nash equilibrium in which members of the same group exert the same effort level. We also derive the exact effort each group will make and show the necessary and sufficient conditions under which each group exerts a positive or zero effort in equilibrium. It turns out that a group may be inactive if its size is relatively large. This is due to the fact that the free-rider incentive within a group becomes more pervasive if its size increases, while at the same time the per-capita reward becomes smaller since each player has to share the prize with more group members (see Esteban and Ray (2001) for a general discussion of these issues in group contests with complete information). It also turns out that the decision of the privately informed large-type group to be active or inactive can be affected by the number of members in the (non-realized) small-type one. Furthermore, the total effort of the groups without private information is larger if the group with private information is less likely to play. Finally, we compare our model with private information to a benchmark model with complete information. We find that the large-type group with private information may alter its decision to be active relative to the benchmark case. Our model can be applied to situations in which groups of incumbent players, who are familiar with each other, face entry of a new group of players who are privately informed about their group size. One can think of the competition for a grant between established and one newly formed group of researchers.

Our analysis is related to four strands of the rent-seeking literature. First, a small but growing number of studies examine contests with private information regarding the valuation of the prize. In particular, Hurley and Shogren (1998a) consider a contest with two players, where one player has private information about his valuation of the prize. Hurley and Shogren (1998b) also study this case and numerically investigate the situation with two-sided private information. Malueg and Yates (2004) analytically investigate the latter case, assuming that the valuation of each player is either high or low, and imposing a simple structure on the probabilities associated with these valuations. Other related papers are Hurley and Shogren (1998c), Wärneryd (2003), Fey (2008) and Ryvkin (2010). Notice that these studies focus on contests between individual players rather than groups of players. We remark that we can easily reformulate our model and results such that one group has private information regarding its valuation of the prize (which can be either high or low) rather than with respect to its number of members. Second, Baik and Lee (2007) investigate asymmetric information in the group contest of Nitzan (1991) with two groups. However, they focus on the case where each group has private information regarding its sharing-rule of the prize. For related studies, see Baik and Lee (2010) and Nitzan and Ueda (2010). Third, a number of papers study whether players (or groups) exert a positive or zero effort in equilibrium. In particular, Stein (2002) considers a contest where players can have possibly different valuations of the prize. He provides the conditions under which a player is active in equilibrium. Note that Stein focuses on a model with individual players and complete information. Ueda (2002) considers the group contest of Nitzan (1991) and derives the conditions under which groups are active in equilibrium. However, that study does not take into account private information. Schoonbeek and Winkel (2006) investigate a contest with individual players, of which one possesses private information about his valuation of the prize, and derive the conditions under which the privately informed player is active in equilibrium. Fourth, a few papers have studied contests with an unknown number of players who each enter the contest randomly, see Münster (2006), Lim and Matros (2009) and Fu *et al.* (2010). Note that these studies focus on contests with individual players and do not consider group contests.

This paper is organized as follows. Section 2 presents the model. Section 3 gives the analysis of the equilibrium. Section 4 concludes. The Appendix contains technical details and proofs.

2 The model

Consider a rent-seeking contest where risk-neutral players compete in $m \geq 2$ groups for a single prize of value $V > 0$ by exerting non-refundable efforts. Every player is in exactly one group. Group 1 has n_S players with probability p and n_L players with probability $1-p$, where $n_L > n_S \geq 1$ and $0 < p < 1$. Hence, we either have a *small-type* or *large-type* group 1. The players of group 1 know the realization of their type, while the players of the other groups only know the distribution. Group $i = 2, \ldots, m$ has $n_i \geq 1$ players, which is publicly known. We assume without loss of generality that $n_2 \leq n_3 \leq \ldots \leq n_m$.

Let t ($t = S, L$) be the realized type of group 1, n_t the number of players in group 1 of type t, $x_{1jt} \geq 0$ the effort of player j ($j = 1, \ldots, n_t$) from group 1 of type t, and $x_{ij} \geq 0$ the effort of player j ($j = 1, \ldots, n_i$) from group $i = 2, 3, \ldots, m$. The corresponding (aggregate) group efforts are denoted by $x_{1t} = \sum_{j=1}^{n_t} x_{1jt}$ and $x_i = \sum_{j=1}^{n_i} x_{ij}$, $i = 2, \ldots, m$, respectively. If group 1 is of type t, the probability that group i wins the prize is given by the Tullock success function

$$q_{it} = \begin{cases} \frac{x_{1t}}{x_{1t}+\sum_{k=2}^{m} x_k}, & \text{for } i = 1, \\ \frac{x_i}{x_{1t}+\sum_{k=2}^{m} x_k}, & \text{for } i = 2, 3, \ldots, m. \end{cases} \tag{1}$$

If $x_{1t} + \sum_{k=2}^{m} x_k = 0$, the probability that group $i = 1, 2, \ldots, m$ wins is $1/m$.

Each player will choose his effort to maximize his expected payoff. If a group wins, all its players will receive an equal share of the prize. Disregarding the case where all group efforts are zero, since that will not occur in equilibrium, the expected payoff of player j ($j = 1, \dots, n_t$) of type t from group 1, π_{1jt}, is

$$\pi_{1jt} = \frac{x_{1t}V}{(x_{1t} + \sum_{k=2}^{m} x_k)n_t} - x_{1jt}, \tag{2}$$

while the expected payoff π_{ij} for player j ($j = 1, \dots, n_i$) from group $i = 2, 3, \dots, m$ is

$$\pi_{ij} = p\frac{x_i V}{(x_{1S} + \sum_{k=2}^{m} x_k)n_i} + (1-p)\frac{x_i V}{(x_{1L} + \sum_{k=2}^{m} x_k)n_i} - x_{ij}. \tag{3}$$

3 The equilibrium

We will investigate the equilibrium properties of our game. Doing so, we focus on pure strategy equilibria in which all players of the same group choose the same effort level. In Section 3.1 we examine the different cases that might hold for the relevant first-order conditions. In Section 3.2 we show that the game has a unique equilibrium. In Section 3.3 we discuss the factors influencing players of group 1 to exert a positive or zero effort in equilibrium. In Section 3.4 we compare our model with private information to a benchmark model with complete information.

3.1 The first-order conditions

An equilibrium in which all players who belong to the same group exert the same effort, denoted by $(\hat{x}_{1S}, \hat{x}_{1L}, \hat{x}_2, \dots, \hat{x}_m)$, is characterized by the following first-order conditions (FOCs), where $\hat{x} = \sum_{k=2}^{m} \hat{x}_k$:

$$\frac{\hat{x}V}{(\hat{x}_{1S} + \hat{x})^2 n_S} \leq 1, \tag{4}$$

$$\frac{\hat{x}V}{(\hat{x}_{1L} + \hat{x})^2 n_L} \leq 1, \tag{5}$$

$$p\frac{(\hat{x}_{1S} + \hat{x} - \hat{x}_i)V}{(\hat{x}_{1S} + \hat{x})^2 n_i} + (1-p)\frac{(\hat{x}_{1L} + \hat{x} - \hat{x}_i)V}{(\hat{x}_{1L} + \hat{x})^2 n_i} \leq 1, \qquad \text{if } i > 1, \tag{6}$$

where (4), (5) and (6) hold with equality if $\hat{x}_{1S} > 0$, $\hat{x}_{1L} > 0$ and $\hat{x}_i > 0$, respectively. Notice in passing that the FOCs are identical to those that correspond to a reformulation of our model in which group 1 has private information regarding its valuation of the prize rather than its number of members. This can be done by defining $V_H = V/n_S$ and $V_L = V/n_L$, i.e. a high and low valuation of the prize, and normalizing the number of members in group 1 to one.

Case	$\hat{x}_{1S}$	$\hat{x}_{1L}$	$\hat{x}$	*Can this yield an equilibrium?*
1	0	0	0	No, for a player in group $i > 1$ playing an infinitesimal ϵ yields expected payoff of $\frac{V}{n_i} - \epsilon$, which is larger than $\frac{V}{mn_i}$, the payoff of playing zero. Similar argument applies to players of group 1.
2	0	0	> 0	No if $m = 2$, since (6) fails. Yes, possibly if $m > 2$.
3	0	> 0	0	No, it implies that (5) holds with equality. While $n_L > n_S$ and $\hat{x}_{1S} = 0$, (4) fails.
4	> 0	0	0	No, (4) fails.
5	0	> 0	> 0	No, it implies that (5) holds with equality. While $n_L > n_S$ and $\hat{x}_{1S} = 0$, (4) fails.
6	> 0	0	> 0	Yes, possibly.
7	> 0	> 0	0	No, (4) and (5) fail.
8	> 0	> 0	> 0	Yes, possibly.

Table 1: Analysis of potential equilibria.

We split the analysis of the FOCs into eight distinct cases based on the decision variables being zero or taking a positive value. In Table 1 we directly show that five of the cases do not yield an equilibrium. From the table we deduct that (6) is always binding. We also note that for $a, b \in \{2, 3, \ldots, m\}$ we have (i) $a \geq b \Leftrightarrow n_a \geq n_b \Leftrightarrow \hat{x}_a \leq \hat{x}_b$, and (ii) $a > b \Leftarrow n_a > n_b \Leftrightarrow \hat{x}_a < \hat{x}_b$. Thus, larger groups make a smaller effort. This is inherent to the free-rider incentive within groups and the fact that the per-capita reward for each player becomes smaller if the number of members in his group increases. Apparently, the effort per player decreases faster than the number of members in the group increases. Further, note that group 2 will always exert a positive effort, and that (6) implies that groups with indices 2 to m with equal size make the same effort.

Let $r \in \{1, 2, \ldots, m-1\}$ be such that (i) $\hat{x}_{r+1} > 0$ and (ii) either $\hat{x}_{r+2} = 0$ or $r + 1 = m$. Note that if $r = m - 1$, the $\hat{x}_{r+2} = 0$ condition does not exist. The variable r represents the number of groups that play a positive effort, *excluding* group 1, the group that possesses private information. For example, suppose there are four groups ($m = 4$)

of which group 1, 2 and 3 play a positive effort, $\hat{x}_{1S} > 0$, $\hat{x}_{1L} > 0$, $\hat{x}_2 > 0$ and $\hat{x}_3 > 0$, while group 4 does not, $\hat{x}_4 = 0$. In this case, $r = 2$. Note also that $\hat{x}_2 \geq \hat{x}_3$.

We consider the three cases that are left in Table 1 one by one. In the Appendix we show for each case that if the FOCs hold, the solution is unique, and we derive the corresponding values of $\hat{x}$ (i.e. the total effort of groups 2 to m) and r. In particular, if subscript $[c]$ denotes case $c \in \{2, 6, 8\}$, we find the following.

Case 2 Assume $m > 2$. If this case holds, we have $\hat{x}_{[2]1S} = 0$, $\hat{x}_{[2]1L} = 0$, $\hat{x}_{[2]i} > 0$ for $2 \leq i \leq r_{[2]} + 1$, and $\hat{x}_{[2]i} = 0$ for $r_{[2]} + 1 < i \leq m$, with

$$\hat{x}_{[2]} = \frac{r_{[2]} - 1}{\sum_{j=2}^{r_{[2]}+1} n_j} V, \tag{7}$$

and

$$r_{[2]} = \max\{i \in 1, 2, \ldots, m-1 \,|\, 1 - \frac{i-1}{\sum_{j=2}^{i+1} n_j} n_{i+1} > 0\}. \tag{8}$$

Case 6 If this case holds, we have $\hat{x}_{[6]1S} = 0$, $\hat{x}_{[6]1L} = 0$, $\hat{x}_{[6]i} > 0$ for $2 \leq i \leq r_{[6]} + 1$, and $\hat{x}_{[6]i} = 0$ for $r_{[6]} + 1 < i \leq m$, with

$$\hat{x}_{[6]} = \left(\frac{r_{[6]} p \sqrt{V n_S} + \sqrt{r_{[6]}^2 p^2 V n_S + 4(p n_S + \sum_{j=2}^{r_{[6]}+1} n_j)(1-p) V (r_{[6]} - 1)}}{2(p n_S + \sum_{j=2}^{r_{[6]}+1} n_j)} \right)^2, \tag{9}$$

and

$$\begin{aligned} r_{[6]} = \max\{i \in 1, 2, \ldots, m-1 \,|\, & p\sqrt{V n_S} \left(\frac{i p \sqrt{V n_S} + \sqrt{i^2 p^2 V n_S + 4(p n_S + \sum_{j=2}^{i+1} n_j)(1-p) V (i-1)}}{2(p n_S + \sum_{j=2}^{i+1} n_j)} \right) \\ & + (1-p)V \\ & - \left(\frac{i p \sqrt{V n_S} + \sqrt{i^2 p^2 V n_S + 4(p n_S + \sum_{j=2}^{i+1} n_j)(1-p) V (i-1)}}{2(p n_S + \sum_{j=2}^{i+1} n_j)} \right)^2 n_{i+1} > 0\}. \end{aligned} \tag{10}$$

Case 8 If this case holds, we have $\hat{x}_{[8]1S} > 0$, $\hat{x}_{[8]1L} > 0$, $\hat{x}_{[8]i} > 0$ for $2 \leq i \leq r_{[8]} + 1$, and $\hat{x}_{[8]i} = 0$ for $r_{[8]} + 1 < i \leq m$, with

$$\hat{x}_{[8]} = \frac{\left(r_{[8]} p \sqrt{V n_S} + r_{[8]} (1-p) \sqrt{V n_L} \right)^2}{\left(p n_S + (1-p) n_L + \sum_{j=2}^{r_{[8]}+1} n_j \right)^2}, \tag{11}$$

and

$$r_{[8]} = \max\{i \in 1, 2, \ldots, m-1 \,|\, pn_S + (1-p)n_L - in_{i+1} + \sum_{j=2}^{i+1} n_j > 0\}. \tag{12}$$

3.2 Main results

We now show that exactly one of the three cases 2, 6 and 8 is true for the game defined by $(m, V, p, n_S, n_L, n_2, \ldots, n_m)$. As a result, our game always has a unique (pure-strategy) equilibrium (in which all players of the same group exert the same effort). First, we introduce the following functions for $i \in \{1, \ldots, m-1\}$:

$$\hat{x}_{[2]}(i) = \frac{i-1}{\sum_{j=2}^{i+1} n_j} V, \tag{13}$$

$$\hat{x}_{[6]}(i) = \left(\frac{ip\sqrt{Vn_S} + \sqrt{i^2p^2Vn_S + 4(pn_S + \sum_{j=2}^{i+1} n_j)(1-p)V(i-1)}}{2(pn_S + \sum_{j=2}^{i+1} n_j)} \right)^2, \tag{14}$$

$$\hat{x}_{[8]}(i) = \frac{\left(ip\sqrt{Vn_S} + i(1-p)\sqrt{Vn_L}\right)^2}{\left(pn_S + (1-p)n_L + \sum_{j=2}^{i+1} n_j\right)^2}. \tag{15}$$

Note that $\hat{x}_{[2]} = \hat{x}_{[2]}(r_{[2]})$, $\hat{x}_{[6]} = \hat{x}_{[6]}(r_{[6]})$ and $\hat{x}_{[8]} = \hat{x}_{[8]}(r_{[8]})$.

Next, we present four lemmas, the proof of which can be found in the Appendix. The first two lemmas give useful properties of the functions $\hat{x}_{[2]}(i)$, $\hat{x}_{[6]}(i)$ and $\hat{x}_{[8]}(i)$.

Lemma 1. *For $i \in \{1, \ldots, m-1\}$ the following holds:*

$$\hat{x}_{[2]}(i) \geq \frac{V}{n_S} \Leftrightarrow (i-1)n_S \geq \sum_{j=2}^{i+1} n_j, \tag{16}$$

$$\hat{x}_{[6]}(i) < \frac{V}{n_S} \Leftrightarrow (i-1)n_S < \sum_{j=2}^{i+1} n_j, \tag{17}$$

$$\hat{x}_{[6]}(i) \geq \frac{V}{n_L} \Leftrightarrow (i-1)\left(p\sqrt{n_Sn_L} + (1-p)n_L\right) \geq pn_S - p\sqrt{n_Sn_L} + \sum_{j=2}^{i+1} n_j, \tag{18}$$

$$\hat{x}_{[8]}(i) < \frac{V}{n_L} \Leftrightarrow (i-1)\left(p\sqrt{n_Sn_L} + (1-p)n_L\right) < pn_S - p\sqrt{n_Sn_L} + \sum_{j=2}^{i+1} n_j. \tag{19}$$

Lemma 2. *The following holds for the game defined by $(m, V, p, n_S, n_L, n_2 \ldots, n_m)$:*

(i) In case 2, the number of active groups of groups 2 *to* m*, represented by* $r_{[2]}$*, is such that it maximizes* $\hat{x}_{[2]}(i)$*:*

$$\hat{x}_{[2]}(i) < \hat{x}_{[2]}(i+1) \quad \textit{for } i \in \{1, \ldots, r_{[2]} - 1\}, \tag{20}$$

$$\hat{x}_{[2]}(i) \geq \hat{x}_{[2]}(i+1) \quad \textit{for } i \in \{r_{[2]}, \ldots, m - 2\}. \tag{21}$$

(ii) In case 6, the number of active groups of groups 2 *to* m*, represented by* $r_{[6]}$*, is such that it maximizes* $\hat{x}_{[6]}(i)$*:*

$$\hat{x}_{[6]}(i) < \hat{x}_{[6]}(i+1) \quad \textit{for } i \in \{1, \ldots, r_{[6]} - 1\}, \tag{22}$$

$$\hat{x}_{[6]}(i) \geq \hat{x}_{[6]}(i+1) \quad \textit{for } i \in \{r_{[6]}, \ldots, m - 2\}. \tag{23}$$

(iii) In case 8, the number of active groups of groups 2 *to* m*, represented by* $r_{[8]}$*, is such that it maximizes* $\hat{x}_{[8]}(i)$*:*

$$\hat{x}_{[8]}(i) < \hat{x}_{[8]}(i+1) \quad \textit{for } i \in \{1, \ldots, r_{[8]} - 1\}, \tag{24}$$

$$\hat{x}_{[8]}(i) \geq \hat{x}_{[8]}(i+1) \quad \textit{for } i \in \{r_{[8]}, \ldots, m - 2\}. \tag{25}$$

The next two lemmas characterize useful properties of cases 2, 6 and 8.

Lemma 3. *The game defined by* $(m, V, p, n_S, n_L, n_2, \ldots, n_m)$ *has an equilibrium in*

(i) case 2 if and only if $\hat{x}_{[2]}(r_{[2]}) \geq \frac{V}{n_S}$*,*

(ii) case 6 if and only if $\frac{V}{n_S} > \hat{x}_{[6]}(r_{[6]}) \geq \frac{V}{n_L}$*,*

(iii) case 8 if and only if $\frac{V}{n_L} > \hat{x}_{[8]}(r_{[8]})$*.*

Lemma 4. *The following holds for the game defined by* $(m, V, p, n_S, n_L, n_2, \ldots, n_m)$*:*

(i) The game cannot have an equilibrium in both case 2 and case 6, i.e.

$$\hat{x}_{[2]}(r_{[2]}) \geq \frac{V}{n_S} \Rightarrow \hat{x}_{[6]}(r_{[6]}) \geq \frac{V}{n_S}. \tag{26}$$

(ii) The game cannot have an equilibrium in both case 2 and case 8, i.e.

$$\hat{x}_{[2]}(r_{[2]}) \geq \frac{V}{n_S} \Rightarrow \hat{x}_{[8]}(r_{[8]}) \geq \frac{V}{n_L}. \tag{27}$$

(iii) The game cannot have an equilibrium in both case 6 and case 8, i.e.

$$\hat{x}_{[8]}(r_{[8]}) < \frac{V}{n_L} \Rightarrow \hat{x}_{[6]}(r_{[6]}) < \frac{V}{n_L}. \tag{28}$$

(iv) The game must have an equilibrium in at least one of the cases 2, 6 and 8, i.e.

$$\hat{x}_{[6]}(r_{[6]}) < \frac{V}{n_L} \Rightarrow \hat{x}_{[8]}(r_{[8]}) < \frac{V}{n_L}, \tag{29}$$

$$\hat{x}_{[6]}(r_{[6]}) \geq \frac{V}{n_S} \Rightarrow \hat{x}_{[2]}(r_{[2]}) \geq \frac{V}{n_S}. \tag{30}$$

Our main proposition directly follows from Lemmas 1 to 4.

Proposition 1. *The game defined by* $(m, V, p, n_S, n_L, n_2, \ldots, n_m)$ *has exactly one (pure-strategy) equilibrium in which players of the same group exert the same effort. It can be expressed as* $(\hat{x}_{1S}, \hat{x}_{1L}, \hat{x}_2, \ldots, \hat{x}_m)$.

Lemma 3 shows that the total equilibrium effort of groups 2 to m is larger in case 2, when both the small-type and large-type group 1 play a zero effort, than in case 6, when there is a chance that group 1 plays, namely if it is the small-type. In turn, the total equilibrium effort of groups 2 to m in case 6 is larger than in case 8, when it is certain that group 1 plays, both when it is a small-type and large-type. This makes sense, as groups 2 to m have a larger chance of obtaining the prize when group 1 does not compete, making them more eager in the contest.

We conclude this section with a numerical example. Consider our game with ten groups ($m = 10$), which all value the prize at $V = 1$. Group 1 has size $n_S = 1$ with probability $p = 0.4$ and size $n_L = 8$ with probability $1 - p = 0.6$. Groups 2 to 10 have the following group sizes: $(3, 4, 6, 9, 9, 10, 10, 14, 16)$. With help of (8), (10) and (12), we calculate the number of groups excluding group 1 that play a positive effort in each case: $r_{[2]} = 3$, $r_{[6]} = 3$ and $r_{[8]} = 3$. Furthermore, by (7), (9) and (11), we compute the total effort of those groups for each case: $\hat{x}_{[2]} \approx 0.154$, $\hat{x}_{[6]} \approx 0.121$ and $\hat{x}_{[8]} \approx 0.119$. Because $\hat{x}_{[2]}$ is smaller than $V/n_S = 1$ and $\hat{x}_{[6]}$ and $\hat{x}_{[8]}$ are both smaller than $V/n_L = 1/8$, only case 8 can yield an equilibrium. It can be verified that in the equilibrium we have $\hat{x}_{1S} \approx 0.226$ and $\hat{x}_{1L} = 0.03$. The equilibrium efforts of groups 2, 3 and 4 are $\hat{x}_2 \approx 0.070$, $\hat{x}_3 \approx 0.047$ and $\hat{x}_4 \approx 0.002$. As $r_{[8]} = 3$, $\hat{x}_5 = \hat{x}_6 = \ldots = \hat{x}_{10} = 0$.

3.3 Factors influencing group 1 to be active or inactive

We will investigate the factors which determine whether (a player of) group 1 is active or inactive, i.e. exerts a positive or zero effort, in equilibrium. The small-type group 1 is always active in equilibrium if $m = 2$. Hence, if we consider this group in this section, we (implicitly) focus on the case with $m > 2$. Let us first examine the impact of n_S and n_L. Notice from (i) of Lemma 3 and (16) that the equilibrium is in case 2 if and only if

$$(r_{[2]} - 1)n_S \geq \sum_{j=2}^{r_{[2]}+1} n_j. \tag{31}$$

Since $r_{[2]}$ is independent of n_S and n_L, we see that the equilibrium is in case 2 if n_S is 'large'. Then both the small-type and large-type group 1 are inactive due to the pervasive free-rider incentive and small per-capita reward in either of these groups (recall that $n_L > n_S$). Next, examine the situation where (31) does not hold, i.e. when n_S is 'small'. Then we are in case 6 or 8 and the small-type group 1 is active in equilibrium. Using (ii) and (iii) of Lemma 3 and the fact that $\hat{x}_{[6]}(r_{[6]})$ does not depend on n_L, we see that we have case 6 if n_L is 'large' and case 8 if n_L is 'small'. In other words, whether the large-type group 1 is (in)active depends on the size of n_L. If n_L is 'large', then this group is inactive because the free-rider incentive is pervasive and the per-capita reward is small. If n_L is 'small', then this group is active for the opposite reasons. In sum, we have case 2 if both n_S and n_L are 'large', case 6 if n_S is 'small' and n_L is 'large', and case 8 if both n_S and n_L are 'small'.

Proceeding, take a situation where the model has an equilibrium in case 8, i.e. in which both types of group 1 are active. Next, let us increase n_S, while keeping n_L fixed (taking into account that $n_L > n_S$). In other words, we aggravate the free-rider incentive and decrease the per-capita reward in the small-type group 1 only. Interestingly, it then might happen for some values of n_S that the large-type group 1 rather than the small-type one becomes inactive. The reason is that the small-type group 1 decreases its effort if n_S becomes larger. As a result, the total effort of groups 2 to m increases, which in turn is unattractive for the large-type group 1. Hence, we can have an indirect effect of the size of the small-type group 1 on the (in)activity of the large-type group 1. This indirect effect can be illustrated if we increase n_S from 1 to 2 in the numerical example of Section 3.1. In that case, the equilibrium falls in case 6 with $r_{[6]} = 3$, $\hat{x}_{1S} = 0.125$ and $\hat{x}_{1L} = 0$. The total effort of groups 2, 3 and 4 has increased to $\hat{x}_{[6]} = 0.132$, with $\hat{x}_2 \approx 0.077$, $\hat{x}_3 \approx 0.052$ and $\hat{x}_4 \approx 0.003$.

Next, consider the impact of $n_2, \ldots, n_m$ and m on the decision of group 1 to be active or inactive in equilibrium. We know that the small-type group 1 is inactive if and only if (31) holds. Using (iii) of Lemma 3 and the negation of (19) with $i = r_{[8]}$, we can obtain the condition such that the large-type group 1 is inactive. Note that $r_{[2]}$ and $r_{[8]}$ are dependent on $n_2, \ldots, n_m$ and m, which makes it generally impossible to determine the impact of these parameters on the decision of group 1 to be (in)active. However, let us focus now on the special case where groups 2 to m have the same size, say n. Then $r_{[2]} = r_{[8]} = m - 1$. As a result, (31) reduces to $f(m) \geq n/n_S$, where $f(m) = (m-2)/(m-1)$, with $f(m)$ increasing in m and $\lim_{m\to\infty} f(m) = 1$. The large-type group 1 is inactive if and only if

$$(m-1)\left(p\sqrt{n_S n_L} + (1-p)n_L - n\right) \geq p n_S + (1-p)n_L. \tag{32}$$

Examining the impact of n, note that (31) and (32) certainly do not hold if n is large enough. Take now a situation where the equilibrium is in case 2, i.e. (31) holds and both types of group 1 are inactive (we must have $m > 2$ in this case). Next, increase n. Then there exist thresholds for n, say $\hat{n}$ and $\tilde{n}$, with $\hat{n} \geq \tilde{n}$, such that: (i) if $n \geq \tilde{n}$, then (31) no longer holds, so the small-type group 1 becomes active (now the equilibrium must be in either case 6 or 8); and (ii) if $n \geq \hat{n}$, then (32) no longer holds, so the large-type group 1 becomes active as well (now the equilibrium must be in case 8). Hence, we see that first the small-type group 1 and next the large-type group 1 switches from inactivity to activity if n becomes large enough. Increases of n correspond to a larger free-rider incentive and smaller per-capita reward in groups 2 to m.

Next, consider the impact of m. Note that (31) does not hold for all $m \geq 2$ if $n/n_S \geq 1$. If $n/n_S < 1$, then (31) certainly will hold if m is large enough. Further, (32) does not hold for all $m \geq 2$ if $n \geq p\sqrt{n_S n_L} + (1-p)n_L$. If $n < p\sqrt{n_S n_L} + (1-p)n_L$, then (32) certainly will hold if m is large enough. Notice that $n_S < p\sqrt{n_S n_L} + (1-p)n_L < n_L$. Now take a situation where the equilibrium is in case 8, i.e. both (31) and (32) do not hold and both types of group 1 are active. Next, increase m. Then there are three possible subcases depending on the size of n:

(a) Suppose $n < n_S$. Then (31) and (32) will hold if m becomes large enough, so both types of group 1 become inactive (then the equilibrium must be in case 2);

(b) Suppose $n_S \leq n < p\sqrt{n_S n_L} + (1-p)n_L$. Then (only) (32) will hold if m becomes large enough, so (only) the large-type group 1 becomes inactive (then the equilibrium must be in case 6);

(c) Suppose $n \geq p\sqrt{n_S n_L} + (1-p)n_L$. Then (31) and (32) do not hold for all $m \geq 2$, so both types of group 1 remain active (the equilibrium remains in case 8).

Note that the size of n in subcases (a) to (c) can be characterized in terms of an increasing free-rider incentive and decreasing per-capita reward in groups 2 to m. For example, in subcase (c), the free-rider incentive and per-capita reward are, respectively, so large and small, that both types of group 1 remain active even if the number of rival groups for group 1 becomes very large.

Finally, we can also examine the impact of parameter p on the decision of group 1 to be (in)active in equilibrium. The corresponding results do not provide much additional insight. Therefore, for brevity, we omit them here.

3.4 Comparison to the case with complete information

In this section we compare the equilibrium behavior of group 1 in our model with private information to that of group 1 in a benchmark model with complete information. We have two variants of the benchmark model, i.e. one in which group 1 has n_S players and one in which it has n_L players. We call these variants the *small-type* and *large-type benchmark* model, respectively. We can easily obtain the equilibrium results of the benchmark model from our previous analysis. In order to see that, take the model with private information and replace the parameters n_S and n_L by, respectively, the variables $\tilde{n}_S$ and $\tilde{n}_L$. We obtain the small-type benchmark model if $\tilde{n}_S = n_S$ and $\tilde{n}_L = n_S$, and the large-type benchmark model if $\tilde{n}_S = n_L$ and $\tilde{n}_L = n_L$. Hence, in the benchmark model we fix both variables at the same value. Note that we did not allow for $n_S = n_L$ in the model with private information. Yet, let us suppose for the moment that $n_S = n_L$ in that model. Referring to Table 1, note that (4) holds with an equality for case 6.

While $n_S = n_L$, $\hat{x}_{1S} > 0$ and $\hat{x}_{1L} = 0$, (5) then fails. Hence, case 6 cannot yield an equilibrium. Note that the results for cases 2 and 8 are still valid if $n_S = n_L$. This also shows that the benchmark model can only yield an equilibrium in cases 2 and 8. The other parameters in the benchmark model, $m, V, p, n_2 \ldots, n_m$, are equal to those in the private information model. Note that p is a superfluous parameter for the benchmark model as the equilibrium is independent of it.

We denote the number of groups besides group 1 that play a positive effort in the benchmark model by $\tilde{r}$. The following proposition is proved in the Appendix.

Proposition 2. *Compare the equilibrium behavior of group 1 in each variant of the benchmark model to that of the corresponding type of group 1 in the private information model. The following holds:*

(i) *Take the small-type benchmark model. Then group 1 in this model is active if and only if the small-type group 1 in the private information model is active.*

(ii) *Take the large-type benchmark model and suppose $\tilde{r}_{[8]} = 1$. Then group 1 in this model is always active, while the large-type group 1 in the model with private information may be inactive.*

(iii) *Take the large-type benchmark model and suppose $\tilde{r}_{[8]} \geq 2$ and $r_{[8]} = 1$. Then group 1 in this model may be active or inactive, while the same holds for the large-type group 1 in the model with private information. Activity or inactivity in one model does not necessarily imply activity of inactivity in the other model.*

(iv) *Take the large-type benchmark model and suppose $\tilde{r}_{[8]} \geq 2$ and $r_{[8]} \geq 2$. If group 1 in this model is active, then the large-type group 1 in the model with private information is active as well, but the reverse does not hold in general.*

Proposition 2 implies that the decision of the players of group 1 to be active or inactive in equilibrium is independent of having private information if the group is small. When the group is large, having private information may change their decision. With one other active group in the benchmark model, having private information gives group 1 the possibility to be inactive in equilibrium, something that is not possible in the benchmark model. If more than one other group is active (both in the benchmark model and the model with private information), group 1 is more often inclined to be active in the model with private information (the reason being that groups 2 to m are less aggressive in that case, i.e. they exert a smaller effort).

4 Conclusion

We have investigated a rent-seeking contest in which players compete in groups for a prize. One group possesses private information about its number of members, which can be either small or large. All other groups have a publicly known number of members. All groups have the same valuation of the prize. We have shown the existence of a unique equilibrium in which all players of the same group exert the same effort, and provided the necessary and sufficient conditions under which a group exerts a positive or zero effort level in equilibrium. We have interpreted the (in)activity of a group in equilibrium in terms of the pervasiveness of the free-rider incentive and per-capita reward for its members. Finally, we have compared our model with private information to a benchmark model with complete information. We have shown that if the group with private information has a small number of members, it is active in the equilibrium of the model with private information if and only if it is active in the equilibrium of the corresponding benchmark model. This does not necessarily hold if the group has a large number of members.

As far as we know, our paper is the first to study private information regarding a group size in a group contest. An interesting generalization could be to allow groups to have different valuations of the prize. Another possible extension could be to let more groups posses private information. We leave these topics for future research.

Appendix: Technical details and proofs

Derivation of (7) and (8)

Considering case 2, assume $m > 2$, $\hat{x}_{[2]1S} = 0$, $\hat{x}_{[2]1L} = 0$ and $\hat{x}_{[2]} > 0$. As defined, $r_{[2]}$ is such that (i) $\hat{x}_{[2]r_{[2]}+1} > 0$ and (ii) either $\hat{x}_{[2]r_{[2]}+2} = 0$ or $r_{[2]} + 1 = m$. The FOCs become

$$\frac{V}{\hat{x}_{[2]}n_S} \leq 1, \tag{A.1}$$

$$\frac{V}{\hat{x}_{[2]}n_L} \leq 1, \tag{A.2}$$

$$\frac{(\hat{x}_{[2]} - \hat{x}_{[2]i})V}{(\hat{x}_{[2]})^2 n_i} = 1, \qquad \text{if } 2 \leq i \leq r_{[2]} + 1, \tag{A.3}$$

$$\frac{V}{\hat{x}_{[2]}n_i} \leq 1, \qquad \text{if } r_{[2]} + 1 < i \leq m. \tag{A.4}$$

From (A.3) we find for $2 \leq i \leq r_{[2]} + 1$ that

$$\hat{x}_{[2]i} = \hat{x}_{[2]} - \frac{\hat{x}^2_{[2]}n_i}{V}, \tag{A.5}$$

hence

$$\hat{x}_{[2]} = \sum_{j=2}^{r_{[2]}+1} \hat{x}_{[2]j} = \frac{r_{[2]} - 1}{\sum_{j=2}^{r_{[2]}+1} n_j} V. \tag{A.6}$$

which equals (7). Using (A.5) and (A.6) we easily obtain (8).

Note that if (A.1) to (A.4) hold, there is a unique solution where $\hat{x}_{[2]1S} = \hat{x}_{[2]1L} = 0$, $\hat{x}_{[2]i}$ can be calculated from (A.5) for $2 \leq i \leq r_{[2]} + 1$, and $\hat{x}_{[2]i} = 0$ for $r_{[2]} + 1 < i \leq m$.

Derivation of (9) and (10)

Examining case 6, assume $\hat{x}_{[6]1S} > 0$, $\hat{x}_{[6]1L} = 0$ and $\hat{x}_{[6]} > 0$. By definition, $r_{[6]}$ is such that (i) $\hat{x}_{[6]r_{[6]}+1} > 0$ and (ii) either $\hat{x}_{[6]r_{[6]}+2} = 0$ or $r_{[6]} + 1 = m$. The FOCs now read

$$\frac{\hat{x}_{[6]}V}{(\hat{x}_{[6]1S} + \hat{x}_{[6]})^2 n_S} = 1, \tag{A.7}$$

$$\frac{V}{\hat{x}_{[6]}n_L} \leq 1, \tag{A.8}$$

$$p\frac{(\hat{x}_{[6]1S} + \hat{x}_{[6]} - \hat{x}_{[6]i})V}{(\hat{x}_{[6]1S} + \hat{x}_{[6]})^2 n_i} + (1-p)\frac{(\hat{x}_{[6]} - \hat{x}_{[6]i})V}{(\hat{x}_{[6]})^2 n_i} = 1, \qquad \text{if } 2 \leq i \leq r_{[6]} + 1, \tag{A.9}$$

$$p\frac{V}{(\hat{x}_{[6]1S} + \hat{x}_{[6]})n_i} + (1-p)\frac{V}{\hat{x}_{[6]}n_i} \leq 1, \qquad \text{if } r_{[6]} + 1 < i \leq m. \tag{A.10}$$

Substituting (A.7) in (A.9), we find for $2 \leq i \leq r_{[6]} + 1$ that

$$\hat{x}_{[6]i} = \frac{p\sqrt{\hat{x}_{[6]} V n_s} + (1-p)V - \hat{x}_{[6]} n_i}{p n_S + (1-p)\frac{V}{\hat{x}_{[6]}}}. \tag{A.11}$$

As $\hat{x}_{[6]} = \sum_{j=2}^{r_{[6]}+1} \hat{x}_{[6]j}$, we can use (A.11) to obtain

$$\hat{x}_{[6]} = \frac{r_{[6]} p\sqrt{\hat{x}_{[6]} V n_S} + r_{[6]}(1-p)V - \hat{x}_{[6]} \sum_{j=2}^{r_{[6]}+1} n_j}{p n_S + (1-p)\frac{V}{\hat{x}_{[6]}}}. \tag{A.12}$$

Solving (A.12) for $\hat{x}_{[6]}$ is equivalent to solving the quadratic equation

$$\left(p n_S + \sum_{j=2}^{r_{[6]}+1} n_j\right) y^2 - \left(r_{[6]} p \sqrt{V n_S}\right) y - (1-p)V(r_{[6]} - 1) = 0, \tag{A.13}$$

where $y = \sqrt{\hat{x}_{[6]}} > 0$. We find

$$\hat{x}_{[6]} = \left(\frac{r_{[6]} p \sqrt{V n_S} + \sqrt{r_{[6]}^2 p^2 V n_S + 4(p n_S + \sum_{j=2}^{r_{[6]}+1} n_j)(1-p)V(r_{[6]} - 1)}}{2(p n_S + \sum_{j=2}^{r_{[6]}+1} n_j)}\right)^2, \tag{A.14}$$

which equals (9). Note that $\hat{x}_{[6]i} > 0$ if and only if the numerator of (A.11) is positive, as the denominator is positive. As a result, we have

$$r_{[6]} = \max\{i \in 1, 2, \ldots, m-1 \,|\, p\sqrt{\hat{x}_{[6]} V n_S} + (1-p)V - \hat{x}_{[6]} n_{i+1} > 0\}. \tag{A.15}$$

Substituting (A.14) in (A.15) gives (10).

Note that if (A.7) to (A.10) hold, there is a unique solution where $\hat{x}_{[6]1S}$ is defined by (A.7), $\hat{x}_{[6]1L} = 0$, $\hat{x}_{[6]i}$ can be derived from (A.11) for $2 \leq i \leq r_{[6]} + 1$, and $\hat{x}_{[6]i} = 0$ for $r_{[6]} + 1 < i \leq m$.

Derivation of (11) and (12)

Considering case 8, assume $\hat{x}_{[8]1S} > 0$, $\hat{x}_{[8]1L} > 0$ and $\hat{x}_{[8]} > 0$. Recall that $r_{[8]}$ is such that (i) $\hat{x}_{[8]r_{[8]}+1} > 0$ and (ii) either $\hat{x}_{[8]r_{[8]}+2} = 0$ or $r_{[8]} + 1 = m$. The FOCs now equal

$$\frac{\hat{x}_{[8]}V}{(\hat{x}_{[8]1S} + \hat{x}_{[8]})^2 n_S} = 1, \tag{A.16}$$

$$\frac{\hat{x}_{[8]}V}{(\hat{x}_{[8]1L} + \hat{x}_{[8]})^2 n_L} = 1, \tag{A.17}$$

$$p\frac{(\hat{x}_{[8]1S} + \hat{x}_{[8]} - \hat{x}_{[8]i})V}{(\hat{x}_{[8]1S} + \hat{x}_{[8]})^2 n_i} + (1-p)\frac{(\hat{x}_{[8]1L} + \hat{x}_{[8]} - \hat{x}_{[8]i})V}{(\hat{x}_{[8]1L} + \hat{x}_{[8]})^2 n_i} = 1, \quad \text{if } 2 \leq i \leq r_{[8]} + 1, \tag{A.18}$$

$$p\frac{V}{(\hat{x}_{[8]1S} + \hat{x}_{[8]})n_i} + (1-p)\frac{V}{(\hat{x}_{[8]1L} + \hat{x}_{[8]})n_i} \leq 1, \quad \text{if } r_{[8]} + 1 < i \leq m. \tag{A.19}$$

Substituting (A.16) and (A.17) in (A.18), we derive for $2 \leq i \leq r_{[8]} + 1$ that

$$\hat{x}_{[8]i} = \frac{p\sqrt{\hat{x}_{[8]}Vn_S} + (1-p)\sqrt{\hat{x}_{[8]}Vn_L} - n_i\hat{x}_{[8]}}{pn_S + (1-p)n_L}. \tag{A.20}$$

Since $\hat{x}_{[8]} = \sum_{j=2}^{r_{[8]}+1} \hat{x}_{[8]j}$, (A.20) yields

$$\hat{x}_{[8]} = \frac{r_{[8]}p\sqrt{\hat{x}_{[8]}Vn_S} + r_{[8]}(1-p)\sqrt{\hat{x}_{[8]}Vn_L} - \sum_{j=2}^{r_{[8]}+1} n_j\hat{x}_{[8]}}{pn_S + (1-p)n_L}. \tag{A.21}$$

Solving (A.21) for $\hat{x}_{[8]}$ is equivalent to solving

$$\left(pn_S + (1-p)n_L + \sum_{j=2}^{r_{[8]}+1} n_j\right) y^2 - \left(r_{[8]}p\sqrt{Vn_S} + r_{[8]}(1-p)\sqrt{Vn_L}\right) y = 0, \tag{A.22}$$

where $y = \sqrt{\hat{x}_{[8]}} > 0$. It follows that

$$\hat{x}_{[8]} = \frac{\left(r_{[8]}p\sqrt{Vn_S} + r_{[8]}(1-p)\sqrt{Vn_L}\right)^2}{\left(pn_S + (1-p)n_L + \sum_{j=2}^{r_{[8]}+1} n_j\right)^2}, \tag{A.23}$$

which coincides with (11). Note that $\hat{x}_{[8]i} > 0$ if and only if the numerator of (A.20) is positive, as the denominator is positive. Thus, we obtain

$$r_{[8]} = \max\{i \in 1, 2, \ldots, m-1 \,|\, p\sqrt{Vn_S} + (1-p)\sqrt{Vn_L} - n_{i+1}\sqrt{\hat{x}_{[8]}} > 0\}. \tag{A.24}$$

Substituting (A.23) in (A.24) gives (12).

Note that if (A.16) to (A.19) hold, there is a unique solution where $\hat{x}_{[8]1S}$ and $\hat{x}_{[8]1L}$ are found by solving (A.16) and (A.17), respectively. The values of $\hat{x}_{[8]i}$ for $2 \leq i \leq r_{[8]}+1$ can be derived from (A.20), while $\hat{x}_{[8]i} = 0$ for $r_{[8]} + 1 < i \leq m$.

Proof of Lemma 1

— Equation (16): Follows directly from (13).

— Equation (17): Observe from (14) that

$$
\begin{aligned}
&\sqrt{\hat{x}_{[6]}(i)} < \sqrt{\frac{V}{n_S}} \\
&\Leftrightarrow \frac{ip\sqrt{Vn_S} + \sqrt{i^2p^2Vn_S + 4(pn_S + \sum_{j=2}^{i+1} n_j)(1-p)V(i-1)}}{2(pn_S + \sum_{j=2}^{i+1} n_j)} < \sqrt{\frac{V}{n_S}} \\
&\Leftrightarrow \frac{ip + \sqrt{i^2p^2 + 4(p + \frac{1}{n_S}\sum_{j=2}^{i+1} n_j)(1-p)(i-1)}}{2(\frac{pn_S}{\sum_{j=2}^{i+1} n_j} + 1)} < \frac{1}{n_S}\sum_{j=2}^{i+1} n_j \\
&\Leftrightarrow i^2p^2 + 4(p + \frac{1}{n_S}\sum_{j=2}^{i+1} n_j)(1-p)(i-1) \\
&\qquad < (2-i)^2p^2 + 4(2-i)\frac{p}{n_S}\sum_{j=2}^{i+1} n_j + 4\left(\frac{1}{n_S}\sum_{j=2}^{i+1} n_j\right)^2 \\
&\Leftrightarrow 4(i-1)p^2 + 4p(1-p)(i-1) + \frac{4}{n_S}\sum_{j=2}^{i+1} n_j(i+p-pi-1) \\
&\qquad < \frac{8p}{n_S}\sum_{j=2}^{i+1} n_j - \frac{4ip}{n_S}\sum_{j=2}^{i+1} n_j + 4\left(\frac{1}{n_S}\sum_{j=2}^{i+1} n_j\right)^2 \\
&\Leftrightarrow 4p(i-1) + \frac{4}{n_S}\sum_{j=2}^{i+1} n_j(i-1) < \frac{4p}{n_S}\sum_{j=2}^{i+1} n_j + 4\left(\frac{1}{n_S}\sum_{j=2}^{i+1} n_j\right)^2 \\
&\Leftrightarrow (i-1)n_S < \sum_{j=2}^{i+1} n_j.
\end{aligned} \tag{A.25}
$$

— Equation (18). Notice that (14) yields

$$
\begin{aligned}
&\sqrt{\hat{x}_{[6]}(i)} \geq \sqrt{\frac{V}{n_L}} \\
&\Leftrightarrow \frac{ip\sqrt{Vn_S} + \sqrt{i^2p^2Vn_S + 4(pn_S + \sum_{j=2}^{i+1} n_j)(1-p)V(i-1)}}{2(pn_S + \sum_{j=2}^{i+1} n_j)} \geq \sqrt{\frac{V}{n_L}} \\
&\Leftrightarrow i^2p^2n_Sn_L + 4n_L(pn_S + \sum_{j=2}^{i+1} n_j)(1-p)(i-1) \geq \left(2(pn_S + \sum_{j=2}^{i+1} n_j) - ip\sqrt{n_Sn_L}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&\Leftrightarrow i^2p^2n_Sn_L + 4n_L(pn_S + \sum_{j=2}^{i+1} n_j)(1-p)(i-1) \\
&\quad \geq 4p^2n_S^2 + 8pn_S\sum_{j=2}^{i+1} n_j + 4(\sum_{j=2}^{i+1} n_j)^2 - 4(pn_S + \sum_{j=2}^{i+1} n_j)ip\sqrt{n_Sn_L} + i^2p^2n_Sn_L \\
&\Leftrightarrow n_L(pn_S + \sum_{j=2}^{i+1} n_j)(1-p)(i-1) \\
&\quad \geq p^2n_S^2 + 2pn_S\sum_{j=2}^{i+1} n_j + (\sum_{j=2}^{i+1} n_j)^2 - (pn_S + \sum_{j=2}^{i+1} n_j)ip\sqrt{n_Sn_L} \\
&\Leftrightarrow n_L(pn_S + \sum_{j=2}^{i+1} n_j)(1-p)(i-1) \\
&\quad \geq \left(pn_S + \sum_{j=2}^{i+1} n_j\right)^2 - (pn_S + \sum_{j=2}^{i+1} n_j)(i-1)p\sqrt{n_Sn_L} - (pn_S + \sum_{j=2}^{i+1} n_j)p\sqrt{n_Sn_L} \\
&\Leftrightarrow n_L(1-p)(i-1) \geq pn_S - (i-1)p\sqrt{n_Sn_L} - p\sqrt{n_Sn_L} + \sum_{j=2}^{i+1} n_j \\
&\Leftrightarrow (i-1)\left(p\sqrt{n_Sn_L} + (1-p)n_L\right) \geq pn_S - p\sqrt{n_Sn_L} + \sum_{j=2}^{i+1} n_j \\
&\Leftrightarrow i-1 \geq \frac{pn_S - p\sqrt{n_Sn_L} + \sum_{j=2}^{i+1} n_j}{p\sqrt{n_Sn_L} + (1-p)n_L}. \qquad \text{(A.26)}
\end{aligned}
$$

— Equation (19): Remark from (15) that

$$
\begin{aligned}
&\sqrt{\hat{x}_{[8]}(i)} < \sqrt{\frac{V}{n_L}} \\
\Leftrightarrow\quad & \frac{ip\sqrt{Vn_S} + i(1-p)\sqrt{Vn_L}}{pn_S + (1-p)n_L + \sum_{j=2}^{i+1} n_j} < \sqrt{\frac{V}{n_L}} \\
\Leftrightarrow\quad & i < \frac{pn_S + (1-p)n_L + \sum_{j=2}^{i+1} n_j}{p\sqrt{n_Sn_L} + (1-p)n_L} \\
\Leftrightarrow\quad & i-1 < \frac{pn_S - p\sqrt{n_Sn_L} + \sum_{j=2}^{i+1} n_j}{p\sqrt{n_Sn_L} + (1-p)n_L}. \qquad \text{(A.27)}
\end{aligned}
$$

□

Proof of Lemma 2

— Part (i): For $i = 2, 3, \ldots, r_{[2]}$, we have by (8) that

$$\begin{aligned} n_{i+1} &< \frac{\sum_{j=2}^{i+1} n_j}{i-1} \\ \Leftrightarrow \quad 1 - \frac{1}{i-1} &< 1 - \frac{n_{i+1}}{\sum_{j=2}^{i+1} n_j} \\ \Leftrightarrow \quad \frac{i-1-1}{\sum_{j=2}^{i+1} n_j - n_{i+1}} &< \frac{i-1}{\sum_{j=2}^{i+1} n_j}, \end{aligned} \tag{A.28}$$

which implies that $\hat{x}_{[2]}(i) < \hat{x}_{[2]}(i+1)$ for $i = 1, 2, \ldots, r_{[2]} - 1$.

For $i = r_{[2]} + 1, r_{[2]} + 2, \ldots, m-1$, we have by (8) that

$$n_{i+1} \geq \frac{\sum_{j=2}^{i+1} n_j}{i-1}, \tag{A.29}$$

which is equivalent to

$$n_{i+2} \geq \frac{\sum_{j=2}^{i+2} n_j}{i}, \qquad i = r_{[2]}, r_{[2]} + 2, \ldots, m-2. \tag{A.30}$$

Thus, for $i = r_{[2]}, r_{[2]} + 1, \ldots, m-2$, we obtain

$$\begin{aligned} \frac{i-1}{\sum_{j=2}^{i+1} n_j} &= \frac{i-1}{\sum_{j=2}^{i+2} n_j - n_{i+2}} \\ &\geq \frac{i-1}{\sum_{j=2}^{i+2} n_j - \frac{\sum_{j=2}^{i+2} n_j}{i}} \\ &= \frac{i-1}{\sum_{j=2}^{i+2} n_j (1 - \frac{1}{i})} \\ &= \frac{i}{\sum_{j=2}^{i+2} n_j}, \end{aligned} \tag{A.31}$$

which proves part (i) for $i = r_{[2]}, r_{[2]} + 1, \ldots, m-2$.

— Part (ii): We define for $i = 1, 2, \ldots, m-1$:

$$a(i) = ip\sqrt{Vn_S}, \tag{A.32}$$

$$b(i) = i^2p^2Vn_S + 4\left(pn_S + \sum_{j=2}^{i+1} n_j\right)(1-p)V(i-1), \tag{A.33}$$

$$c(i) = 2\left(pn_S + \sum_{j=2}^{i+1} n_j\right), \tag{A.34}$$

$$f(i) = \frac{a(i) + \sqrt{b(i)}}{c(i)} = \sqrt{\hat{x}_{[6]}(i)}. \tag{A.35}$$

For $i = 2, 3, \ldots, m-1$, we have

$$\begin{aligned} f(i-1) &= \frac{a(i-1) + \sqrt{b(i-1)}}{c(i-1)} \\ &= \frac{a(i-1)}{c(i-1)} + \sqrt{\frac{a(i-1)^2}{\left(c(i-1)\right)^2} + \frac{2(1-p)V(i-2)}{c(i-1)}} \\ &= \frac{\frac{i-1}{i}a(i)}{c(i) - 2n_{i+1}} + \sqrt{\frac{\left(\frac{i-1}{i}\right)^2 a(i)^2}{\left(c(i) - 2n_{i+1}\right)^2} + \frac{2(1-p)V(i-2)}{c(i) - 2n_{i+1}}}. \qquad \text{(A.36)} \end{aligned}$$

From (A.15) we know for $i = r_{[6]} + 1, \ldots, m-1$, that

$$n_{i+1} \geq p\sqrt{Vn_S}\frac{c(i)}{a(i) + \sqrt{b(i)}} + (1-p)V\frac{\left(c(i)\right)^2}{\left(a(i) + \sqrt{b(i)}\right)^2}. \qquad \text{(A.37)}$$

Using (A.37), for $i = r_{[6]} + 1, \ldots, m-1$, we find

$$\begin{aligned} c(i) - 2n_{i+1} &\leq c(i)\left(1 - \frac{2p\sqrt{Vn_S}}{a(i) + \sqrt{b(i)}} - \frac{2(1-p)Vc(i)}{\left(a(i) + \sqrt{b(i)}\right)^2}\right) \qquad \text{(A.38)} \\ &= c(i)\left(1 - \frac{2\frac{\left(a(i)\right)^2}{i} + 2\frac{a(i)}{i}\sqrt{b(i)} + 2(1-p)Vc(i)}{\left(a(i)\right)^2 + 2a(i)\sqrt{b(i)} + b(i)}\right) \\ &= c(i)\left(1 - \frac{1}{i}\left(\frac{2\left(a(i)\right)^2 + 2a(i)\sqrt{b(i)} + 2(1-p)Vc(i)i}{2\left(a(i)\right)^2 + 2a(i)\sqrt{b(i)} + 2(1-p)V(i-1)c(i)}\right)\right) \\ &= c(i)\left(1 - \frac{1}{i}\left(1 + \frac{2(1-p)Vc(i)}{2\left(a(i)\right)^2 + 2a(i)\sqrt{b(i)} + 2(1-p)V(i-1)c(i)}\right)\right) \\ &= c(i)\left(1 - \frac{1}{i} - \frac{2(1-p)Vc(i)}{i\left(a(i) + \sqrt{b(i)}\right)^2}\right). \qquad \text{(A.39)} \end{aligned}$$

Substituting (A.39) into (A.36) yields for $i = r_{[6]} + 1, \ldots, m-1$, that

$$f(i-1) \geq \frac{\frac{i-1}{i}a(i) + \sqrt{\left(\frac{i-1}{i}\right)^2 a(i)^2 + 2(1-p)Vc(i)\left(i - 3 + \frac{2}{i} - \frac{2(1-p)V(i-2)c(i)}{i\left(a(i)+\sqrt{b(i)}\right)^2}\right)}}{c(i)\left(1 - \frac{1}{i} - \frac{2(1-p)Vc(i)}{i\left(a(i)+\sqrt{b(i)}\right)^2}\right)}. \qquad \text{(A.40)}$$

We focus for the moment on $2(1-p)Vc(i)$:

$$\begin{aligned}
2(1-p)Vc(i) &= 2(1-p)Vc(i) + \frac{\left(a(i)\right)^2}{i} - \frac{\left(a(i)\right)^2}{i} + \frac{2(1-p)Vc(i)}{i} - \frac{2(1-p)Vc(i)}{i} \\
&= \frac{b(i)}{i} - \frac{\left(a(i)\right)^2}{i} + \frac{2(1-p)Vc(i)}{i} \\
&= \frac{b(i)}{i} - \frac{\left(a(i)\right)^2}{i} + \frac{2(1-p)V(i-1)c(i)}{i^2} + \frac{2(1-p)Vc(i)}{i^2} \\
&= \frac{b(i)}{i} - \frac{\left(a(i)\right)^2}{i} + \frac{b(i)}{i^2} - \frac{\left(a(i)\right)^2}{i^2} + \frac{2(1-p)Vc(i)}{i^2} \\
&= 2\frac{b(i)}{i} - \frac{b(i)}{i} - \frac{\left(a(i)\right)^2}{i} + 2\frac{b(i)}{i^2} - \frac{b(i)}{i^2} - \frac{\left(a(i)\right)^2}{i^2} + \frac{2(1-p)Vc(i)}{i^2} \\
&\quad + 2\frac{a(i)\sqrt{b(i)}}{i} - 2\frac{a(i)\sqrt{b(i)}}{i} + 2\frac{a(i)\sqrt{b(i)}}{i^2} - 2\frac{a(i)\sqrt{b(i)}}{i^2} \\
&= -\left(\frac{\left(a(i)\right)^2}{i} + \frac{\left(a(i)\right)^2}{i^2} + 2\frac{a(i)\sqrt{b(i)}}{i} + 2\frac{a(i)\sqrt{b(i)}}{i^2} + \frac{b(i)}{i} + \frac{b(i)}{i^2}\right) \\
&\quad + 2\frac{a(i)\sqrt{b(i)}}{i} + 2\frac{a(i)\sqrt{b(i)}}{i^2} + 2\frac{b(i)}{i} + 2\frac{b(i)}{i^2} + \frac{2(1-p)Vc(i)}{i^2} \\
&= -\left(\frac{i+1}{i^2}\right)\left(\left(a(i)\right)^2 + 2a(i)\sqrt{b(i)} + b(i)\right) \\
&\quad + \frac{2}{i}\left(\frac{i+1}{i}\right)a(i)\sqrt{b(i)} + \frac{2}{i}\left(\frac{i+1}{i}\right)b(i) + \frac{2(1-p)Vc(i)}{i^2} \\
&= -\left(\frac{i+1}{i^2}\right)\left(a(i) + \sqrt{b(i)}\right)^2 + \frac{2}{i}\left(\frac{i+1}{i}\right)\sqrt{b(i)}\left(a(i) + \sqrt{b(i)}\right) \\
&\quad + \frac{2(1-p)Vc(i)}{i^2}.
\end{aligned} \tag{A.41}$$

Substituting (A.41) into the square root in the numerator of (A.40) yields

$$\begin{aligned}
&\left(\frac{i-1}{i}\right)^2 a(i)^2 + 2(1-p)Vc(i) \\
&\times \left(i - 3 + \frac{2}{i} + \frac{(i+1)(i-2)}{i^3} - \frac{2(i+1)(i-2)\sqrt{b(i)}}{i^3\left(a(i) + \sqrt{b(i)}\right)} - \frac{2(1-p)V(i-2)c(i)}{i^3\left(a(i) + \sqrt{b(i)}\right)^2}\right) \\
= &\left(\frac{i-1}{i}\right)^2 a(i)^2 + 2(1-p)Vc(i) \\
&\times \left(i - 3 + \frac{2}{i} + \frac{1}{i} - \frac{1}{i^2} - \frac{2}{i^3} - \frac{2(i+1)(i-2)\sqrt{b(i)}}{i^3\left(a(i) + \sqrt{b(i)}\right)} - \frac{2(1-p)V(i-2)c(i)}{i^3\left(a(i) + \sqrt{b(i)}\right)^2}\right)
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{i-1}{i}\right)^2 a(i)^2 + \left(\frac{i-1}{i}\right)^2 2(1-p)V(i-1)c(i) \\
&\quad + 2(1-p)Vc(i)\left(-\frac{2(i-1)\sqrt{b(i)}}{i^2\left(a(i)+\sqrt{b(i)}\right)} + \frac{2(1-p)Vc(i)}{i^2\left(a(i)+\sqrt{b(i)}\right)^2}\right) \\
&\quad + \frac{4}{i^3}(1-p)Vc(i)\left(\frac{2\sqrt{b(i)}}{\left(a(i)+\sqrt{b(i)}\right)} - \frac{2(1-p)V(i-1)c(i)}{\left(a(i)+\sqrt{b(i)}\right)^2} - 1\right) \\
&= \left(\frac{i-1}{i}\right)^2 b(i) - 4\left(\frac{i-1}{i^2}\right)\sqrt{b(i)}\frac{(1-p)Vc(i)}{a(i)+\sqrt{b(i)}} + \frac{4}{i^2}\left(\frac{(1-p)Vc(i)}{a(i)+\sqrt{b(i)}}\right)^2 \\
&\quad + \frac{4}{i^3}(1-p)Vc(i)\left(\frac{2\sqrt{b(i)}}{\left(a(i)+\sqrt{b(i)}\right)} - \frac{2(1-p)V(i-1)c(i)}{\left(a(i)+\sqrt{b(i)}\right)^2} - 1\right) \\
&= \left(\frac{i-1}{i}\right)^2 b(i) - 4\left(\frac{i-1}{i^2}\right)\sqrt{b(i)}\frac{(1-p)Vc(i)}{a(i)+\sqrt{b(i)}} + \frac{4}{i^2}\left(\frac{(1-p)Vc(i)}{a(i)+\sqrt{b(i)}}\right)^2 \\
&\quad + \frac{4(1-p)Vc(i)}{i^3\left(a(i)+\sqrt{b(i)}\right)^2} \\
&\quad \times \left(2\sqrt{b(i)}a(i) + 2b(i) - 2(1-p)V(i-1)c(i) - \left(a(i)\right)^2 - 2a(i)\sqrt{b(i)} - b(i)\right) \\
&= \left(\frac{i-1}{i}\right)^2 b(i) - \frac{4}{i}\left(\frac{i-1}{i}\right)\sqrt{b(i)}\frac{(1-p)Vc(i)}{a(i)+\sqrt{b(i)}} + \frac{4}{i^2}\left(\frac{(1-p)Vc(i)}{a(i)+\sqrt{b(i)}}\right)^2 \\
&= \left(\left(\frac{i-1}{i}\right)\sqrt{b(i)} - \frac{2(1-p)Vc(i)}{i\left(a(i)+\sqrt{b(i)}\right)}\right)^2 . \qquad \text{(A.42)}
\end{aligned}$$

We can substitute (A.42) into (A.40) to obtain for $i = r_{[6]} + 1, \ldots, m-1$, that

$$\begin{aligned}
f(i-1) &\geq \frac{\left(\frac{i-1}{i}\right)a(i) + \sqrt{\left(\left(\frac{i-1}{i}\right)\sqrt{b(i)} - \frac{2(1-p)Vc(i)}{i\left(a(i)+\sqrt{b(i)}\right)}\right)^2}}{c(i)\left(1 - \frac{1}{i} - \frac{2(1-p)Vc(i)}{i\left(a(i)+\sqrt{b(i)}\right)^2}\right)} \qquad \text{(A.43)} \\
&= \frac{\left(\frac{i-1}{i}\right)a(i) + \left(\frac{i-1}{i}\right)\sqrt{b(i)} - \frac{2(1-p)Vc(i)}{i\left(a(i)+\sqrt{b(i)}\right)}}{c(i)\left(1 - \frac{1}{i} - \frac{2(1-p)Vc(i)}{i\left(a(i)+\sqrt{b(i)}\right)^2}\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(\frac{i-1}{i}\right) a(i) + \left(\frac{i-1}{i}\right) \sqrt{b(i)} - \left(a(i) + \sqrt{b(i)}\right) \frac{2(1-p)Vc(i)}{i\left(a(i)+\sqrt{b(i)}\right)^2}}{c(i) \left(1 - \frac{1}{i} - \frac{2(1-p)Vc(i)}{i\left(a(i)+\sqrt{b(i)}\right)^2}\right)} \\
&= f(i), \qquad \text{(A.44)}
\end{aligned}
$$

which shows that $\hat{x}_{[6]}(i) \geq \hat{x}_{[6]}(i+1)$ for $i = r_{[6]}, \dots, m-2$.

From (A.15) we know for $i = 2, 3, \dots, r_{[6]}$, that

$$
n_{i+1} < p\sqrt{Vn_S}\frac{c(i)}{a(i) + \sqrt{b(i)}} + (1-p)V\frac{(c(i))^2}{\left(a(i) + \sqrt{b(i)}\right)^2}. \qquad \text{(A.45)}
$$

Following the rest of the proof above, the inequality signs in equations (A.38), (A.40) and (A.43) change to $>$, $<$ and $<$, respectively. This proves part (ii) for $i = 1, \dots, r_{[6]} - 1$.
— Part (iii): For $i = 2, 3, \dots, r_{[8]}$, we know from (12) that

$$
n_{i+1} < \frac{\sum_{j=2}^{i+1} n_j}{i} + \frac{1}{i}\left(pn_S + (1-p)n_L\right). \qquad \text{(A.46)}
$$

Using (A.46), we know for $i = 2, 3, \dots, r_{[8]}$, that

$$
\begin{aligned}
&\frac{(i-1)p\sqrt{Vn_S} + (i-1)(1-p)\sqrt{Vn_L}}{pn_S + (1-p)n_L + \sum_{j=2}^{i} n_j} \\
&= \frac{\left(1 - \frac{1}{i}\right)\left(ip\sqrt{Vn_S} + i(1-p)\sqrt{Vn_L}\right)}{pn_S + (1-p)n_L + \sum_{j=2}^{i+1} n_j - n_{i+1}} \\
&< \frac{\left(1 - \frac{1}{i}\right)\left(ip\sqrt{Vn_S} + i(1-p)\sqrt{Vn_L}\right)}{pn_S + (1-p)n_L + \sum_{j=2}^{i+1} n_j - \frac{\sum_{j=2}^{i+1} n_j}{i} - \frac{1}{i}\left(pn_S + (1-p)n_L\right)} \\
&= \frac{ip\sqrt{Vn_S} + i(1-p)\sqrt{Vn_L}}{pn_S + (1-p)n_L + \sum_{j=2}^{i+1} n_j}, \qquad \text{(A.47)}
\end{aligned}
$$

which implies that $\hat{x}_{[8]}(i) < \hat{x}_{[8]}(i+1)$ for $i = 1, 2, \dots, r_{[8]} - 1$.

Similarly, for $i = r_{[8]} + 1, \dots, m-1$, we know from (12) that

$$
n_{i+1} \geq \frac{\sum_{j=2}^{i+1} n_j}{i} + \frac{1}{i}\left(pn_S + (1-p)n_L\right). \qquad \text{(A.48)}
$$

Using (A.48), we know for $i = r_{[8]} + 1, \dots, m-1$, that

$$
\frac{(i-1)p\sqrt{Vn_S} + (i-1)(1-p)\sqrt{Vn_L}}{pn_S + (1-p)n_L + \sum_{j=2}^{i} n_j}
$$

$$
\begin{aligned}
&= \frac{\left(1-\frac{1}{i}\right)\left(ip\sqrt{Vn_S}+i(1-p)\sqrt{Vn_L}\right)}{pn_S+(1-p)n_L+\sum_{j=2}^{i+1}n_j-n_{i+1}} \\
&\geq \frac{\left(1-\frac{1}{i}\right)\left(ip\sqrt{Vn_S}+i(1-p)\sqrt{Vn_L}\right)}{pn_S+(1-p)n_L+\sum_{j=2}^{i+1}n_j-\frac{\sum_{j=2}^{i+1}n_j}{i}-\frac{1}{i}\left(pn_S+(1-p)n_L\right)} \\
&= \frac{ip\sqrt{Vn_S}+i(1-p)\sqrt{Vn_L}}{pn_S+(1-p)n_L+\sum_{j=2}^{i+1}n_j}, \qquad \text{(A.49)}
\end{aligned}
$$

which gives part (iii) for $i = r_{[8]}, \dots, m-2$. ☐

Proof of Lemma 3

For each of the three cases 2, 6 and 8 we have that, when the associated FOCs hold, a unique solution is defined. If for a case not all four FOCs hold, the case cannot yield an equilibrium. In each case the third and fourth FOC — i.e. (A.3) and (A.4) for case 2, (A.9) and (A.10) for case 6, and (A.18) and (A.19) for case 8 — *always* hold, as they are implied by the selection of $r_{[2]}$ in (8), $r_{[6]}$ in (10), and $r_{[8]}$ in (12), respectively. This directly follows for the third FOC of each case. We give a proof of the claim for the fourth FOC in all three cases.

For case 2, note that (8) implies for $i = r_{[2]}+1, \dots, m-1$ that $n_{i+1} \geq V/\hat{x}_{[2](i)}$, which by (i) of Lemma 2 gives $n_{i+1} \geq V/\hat{x}_{[2]}(r_{[2]})$, which is equivalent to (A.4). For case 6, note that (A.15) implies for $i = r_{[6]}+1, \dots, m-1$ that

$$n_{i+1} \geq p\sqrt{\frac{Vn_S}{\hat{x}_{[6]}(i)}}+(1-p)\frac{V}{\hat{x}_{[6]}(i)}, \qquad \text{(A.50)}$$

which by (ii) of Lemma 2 gives

$$n_{i+1} \geq p\sqrt{\frac{Vn_S}{\hat{x}_{[6]}(r_{[6]})}}+(1-p)\frac{V}{\hat{x}_{[6]}(r_{[6]})}. \qquad \text{(A.51)}$$

Remark that (A.51) is equivalent to (A.10) if you substitute in (A.7). For case 8, notice that (A.24) implies for $i = r_{[8]}+1, \dots, m-1$ that

$$n_{i+1} \geq \frac{p\sqrt{Vn_S}+(1-p)\sqrt{Vn_L}}{\sqrt{\hat{x}_{[8]}(i)}}, \qquad \text{(A.52)}$$

which by (iii) of Lemma 2 learns that

$$n_{i+1} \geq p\frac{\sqrt{Vn_S}}{\sqrt{\hat{x}_{[8]}(r_{[8]})}}+(1-p)\frac{\sqrt{Vn_L}}{\sqrt{\hat{x}_{[8]}(r_{[8]})}}. \qquad \text{(A.53)}$$

We see that (A.53) is equivalent to (A.19) if you substitute in (A.16) and (A.17).

Finally, considering the two FOCs of each case that do not hold for all parameter values — i.e. (A.1) and (A.2) for case 2, (A.7) and (A.8) for case 6, and (A.16) and (A.17) for case 8 — we can easily complete the proof of the lemma. ☐

Proof of Lemma 4

— Part (i): If we have an equilibrium in case 2, (16) and part (i) of Lemma 3 give

$$(r_{[2]}-1)n_S \geq \sum_{j=1}^{r_{[2]}+1} n_j, \tag{A.54}$$

which by (17) implies $\hat{x}_{[6]}(r_{[2]}) \geq V/n_S$. Using part (ii) of Lemma 2, this gives $\hat{x}_{[6]}(r_{[6]}) \geq V/n_S$. As this last implication is in contradiction with an equilibrium for case 6, we cannot have that cases 2 and 6 yield an equilibrium at the same time.

— Part (ii): When we have an equilibrium in case 8, we have by part (iii) of Lemma 2 and part (iii) of Lemma 3 that $x_{[8]}(r_{[2]}) < V/n_L$. By (19) we then have

$$\begin{aligned}
& (r_{[2]}-1)\left(p\sqrt{n_S n_L}+(1-p)n_L\right) < pn_S - p\sqrt{n_S n_L} + \sum_{j=2}^{r_{[2]}+1} n_j \\
\Leftrightarrow\quad & r_{[2]}\left(p\sqrt{n_S n_L}+(1-p)n_L\right) < pn_S + (1-p)n_L + \sum_{j=2}^{r_{[2]}+1} n_j \\
\Rightarrow\quad & r_{[2]}\left(pn_S+(1-p)n_L\right) < pn_S + (1-p)n_L + \sum_{j=2}^{r_{[2]}+1} n_j \\
\Leftrightarrow\quad & (r_{[2]}-1)\left(pn_S+(1-p)n_L\right) < \sum_{j=2}^{r_{[2]}+1} n_j \\
\Rightarrow\quad & (r_{[2]}-1)n_S < \sum_{j=2}^{r_{[2]}+1} n_j. \qquad\qquad (A.55)
\end{aligned}$$

Note that (A.55) implies by (16) that $\hat{x}_{[2]}(r_{[2]}) < V/n_S$. As this last implication is in contradiction with an equilibrium for case 2, we cannot have that cases 2 and 8 yield an equilibrium at the same time.

— Part (iii): When we have an equilibrium in case 8, we have by part (ii) of Lemma 2 and part (iii) of Lemma 3 that $x_{[8]}(r_{[6]}) < V/n_L$. Then (19) yields

$$(r_{[6]} - 1)\left(p\sqrt{n_S n_L} + (1-p)n_L\right) < pn_S - p\sqrt{n_S n_L} + \sum_{j=2}^{r_{[6]}+1} n_j, \tag{A.56}$$

which by (18) implies $\hat{x}_{[6]}(r_{[6]}) < V/n_L$. As this last implication is in contradiction with an equilibrium for case 6, we cannot have that cases 6 and 8 yield an equilibrium at the same time.

— Part (iv): Assume case 6 does not yield an equilibrium. Then either $\hat{x}_{[6]}(r_{[6]}) \geq V/n_S$ or $\hat{x}_{[6]}(r_{[6]}) < V/n_L$. For the case $\hat{x}_{[6]}(r_{[6]}) \geq V/n_S$, note that (17) implies

$$(r_{[6]} - 1)n_S \geq \sum_{j=2}^{r_{[6]}+1} n_j. \tag{A.57}$$

Using (13) and part (i) of Lemma 2 we find that

$$(r_{[2]} - 1)n_S \geq \sum_{j=2}^{r_{[2]}+1} n_j. \tag{A.58}$$

Applying (16) to (A.58), we know that case 2 yields an equilibrium.

For the case $\hat{x}_{[6]}(r_{[6]}) < V/n_L$, part (ii) of Lemma 2 implies that $\hat{x}_{[6]}(r_{[8]}) < V/n_L$. From (18) and (19) we know that case 8 yields an equilibrium. □

Proof of Proposition 2

Take the small-type benchmark model, i.e. $\tilde{n}_S = \tilde{n}_L = n_S$. The condition that group 1 in this model is active (case 6 or case 8) is the negation of (16) with $i = \tilde{r}_{[2]}$. Similarly, the condition that the small-type group 1 in the model with private information is active is the negation of (16) with $i = r_{[2]}$. Since (8) — which is the function defining $\tilde{r}_{[2]}$ and $r_{[2]}$ — is independent of n_S and n_L, group 1 will be active in the small-type benchmark model if and only if the small-type group 1 is active in the private information model. This proves part (i).

For the rest of this proof, take the large-type benchmark model, i.e. $\tilde{n}_S = \tilde{n}_L = n_L$. Group 1 in this model will be active when (use (19))

$$(\tilde{r}_{[8]} - 1)n_L < \sum_{j=2}^{\tilde{r}_{[8]}+1} n_j. \tag{A.59}$$

We can derive from (19) when the large-type group 1 is active (case 8) in the private information model. Note that (12) implies that $\tilde{r}_{[8]} \geq r_{[8]}$. Hence, if $\tilde{r}_{[8]} = 1$, then $r_{[8]} = 1$ and group 1 will always be active in the large-type benchmark model, while the large-type group 1 can be inactive in the model with private information. Note that this result *a fortiori* holds for the two-group model ($n = 2$ and therefore $r_{[8]} = 1$ and $\tilde{r}_{[8]} = 1$). This proves part (ii).

If $\tilde{r}_{[8]} \geq 2$ and $r_{[8]} = 1$, we compare (A.59) to (19) with $i = r_{[8]} = 1$ and conclude that both group 1 in the large-type benchmark model and the large-type group 1 in the private information model can be active and inactive, without one model being (in)active necessarily implying (in)activity for the other. This proves part (iii).

Finally, we examine the situation with $\tilde{r}_{[8]} \geq 2$ and $r_{[8]} \geq 2$. By (12), we have for the large-type benchmark model that

$$\begin{aligned}
& \frac{\sum_{j=2}^{\tilde{r}_{[8]}+1} n_j}{\tilde{r}_{[8]}} + \frac{n_L}{\tilde{r}_{[8]}} - n_{\tilde{r}_{[8]}+1} > 0 \\
\Leftrightarrow \quad & 1 + \frac{n_L}{\sum_{j=2}^{\tilde{r}_{[8]}+1} n_j} - n_{\tilde{r}_{[8]}+1} \frac{\tilde{r}_{[8]}}{\sum_{j=2}^{\tilde{r}_{[8]}+1} n_j} > 0 \\
\Leftrightarrow \quad & 1 - \frac{\tilde{r}_{[8]} - 1}{\sum_{j=2}^{\tilde{r}_{[8]}+1} n_j} n_{\tilde{r}_{[8]}+1} - \frac{n_{\tilde{r}_{[8]}+1}}{\sum_{j=2}^{\tilde{r}_{[8]}+1} n_j} + \frac{n_L}{\sum_{j=2}^{\tilde{r}_{[8]}+1} n_j} > 0 \\
\Rightarrow \quad & 1 - \frac{\tilde{r}_{[8]} - 1}{\sum_{j=2}^{\tilde{r}_{[8]}+1} n_j} n_{\tilde{r}_{[8]}+1} - \frac{n_{\tilde{r}_{[8]}+1}}{\sum_{j=2}^{\tilde{r}_{[8]}+1} n_j} + \frac{1}{\tilde{r}_{[8]} - 1} > 0, \qquad \text{(A.60)}
\end{aligned}$$

where (A.60) is due to (A.59). Suppose now that $\tilde{r}_{[8]} > \tilde{r}_{[2]}$. Then (8) implies that

$$\frac{n_{\tilde{r}_{[8]}+1}}{\sum_{j=2}^{\tilde{r}_{[8]}+1} n_j} \geq \frac{1}{\tilde{r}_{[8]} - 1}. \qquad \text{(A.61)}$$

If we substitute (A.61) in (A.60) and compare the result with (8), we obtain $\tilde{r}_{[2]} \geq \tilde{r}_{[8]}$, which gives a contradiction. Therefore, we must have $\tilde{r}_{[2]} \geq \tilde{r}_{[8]}$. Note furthermore, that by (8) we have $\tilde{r}_{[2]} = r_{[2]}$. Summarizing, we obtain $r_{[2]} = \tilde{r}_{[2]} \geq \tilde{r}_{[8]} \geq r_{[8]}$. By part (i) of Lemma 2 and (A.59), we now know that

$$\frac{1}{n_L} > \frac{\tilde{r}_{[8]} - 1}{\sum_{j=2}^{\tilde{r}_{[8]}+1} n_j} \Rightarrow \frac{1}{n_L} > \frac{r_{[8]} - 1}{\sum_{j=2}^{r_{[8]}+1} n_j}. \qquad \text{(A.62)}$$

We can derive from (19) that the large-type group 1 in the private information model is active when

$$\begin{aligned}\sum_{j=2}^{r_{[8]}+1} n_j &> (r_{[8]}-1)\left(p\sqrt{n_S n_L}+(1-p)n_L\right)-pn_S+p\sqrt{n_S n_L}\\ &=(r_{[8]}-1)p\sqrt{n_S n_L}+(r_{[8]}-1)(1-p)n_L-pn_S+p\sqrt{n_S n_L}\\ &=r_{[8]}p\sqrt{n_S n_L}+(r_{[8]}-1)(1-p)n_L-pn_S\\ &=pn_L(r_{[8]}\sqrt{\frac{n_S}{n_L}}-\frac{n_S}{n_L})+(r_{[8]}-1)(1-p)n_L. \qquad \text{(A.63)}\end{aligned}$$

Furthermore, because $0 < n_S < n_L$, we have for $r_{[8]} \geq 2$ that

$$\begin{aligned} &\left(\sqrt{\frac{n_S}{n_L}}-1\right)^2 > 0\\ \Leftrightarrow\quad & 2 > \frac{1-\frac{n_S}{n_L}}{1-\sqrt{\frac{n_S}{n_L}}}\\ \Rightarrow\quad & r_{[8]} > \frac{1-\frac{n_S}{n_L}}{1-\sqrt{\frac{n_S}{n_L}}}\\ \Leftrightarrow\quad & r_{[8]}-1 > r_{[8]}\sqrt{\frac{n_S}{n_L}}-\frac{n_S}{n_L}. \qquad \text{(A.64)}\end{aligned}$$

By (A.64) and (A.62), we know that if group 1 in the large-type benchmark model is active in case $r_{[8]} \geq 2$, then

$$pn_L(r_{[8]}\sqrt{\frac{n_S}{n_L}}-\frac{n_S}{n_L})+(r_{[8]}-1)(1-p)n_L < (r_{[8]}-1)n_L < \sum_{j=2}^{r_{[8]}+1} n_j. \qquad \text{(A.65)}$$

Comparing (A.65) to (A.63), we conclude that if group 1 is active in the large-type benchmark model, the large-type group 1 is also active in the model with private information. Note that the reverse does not hold in general. This proves part (iv). ☐

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