MULTI-STAGE PAIRWISE ELIMINATION CONTESTS WITH HETEROGENEOUS AGENTS*

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Abstract

This article presents a solution technique for pairwise elimination contests with heterogeneous players. Players differ in terms of their effort productivities, which are common-knowledge among participants of the contest. It is shown under which conditions a subgame perfect Nash equilibrium exists when a Tullock contest success function is used. Moreover, the equilibrium solution is derived analytically for the special case of a lottery CSF, and approximately for the remaining cases. A distinct feature of multi-stage pairwise elimination contests with heterogeneous agents is that continuation values in early stages become endogenous due to feedback effects across different branches of the game. Those feedback effects are analyzed in some detail, as well as several other properties of the equilibrium solution.

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1 Introduction

Contest models are used to describe strategic interactions between agents in many different settings, including diverse areas such as war, rent-seeking competitions, or R&D tournaments. Due to the impressive variety of possible applications, many different contest structures are have been considered in the literature already. One of the most prominent one is the multi-stage pairwise elimination contest, which is sometimes also referred to as knock-out tournament. This contest structure is probably best known from sports: Disciplines like baseball, boxing, hockey, soccer, tennis, or even chess make use of this structure at least in later stages of the competition, in the so-called "playoff" stage. However, the structural feature of subsequent elimination is relevant in many other fields as well: In Personnel Economics, for example, where promotion tournaments within firms are usally modeled as elimination contests, or in Political Sciences, where multi-stage election campaigns like the one for US presidency have this structural feature: Candidates first compete against competitors in their own party, and only the winner of this competition meets the opponent from the rival party in the second stage.

In this paper, I present a solution technique for multi-stage pairwise elimination contests with heterogeneous players, assuming that player types are common-knowledge among participants of the contest. I shown under which conditions a subgame perfect Nash equilibrium exists when a general Tullock contest success function (CSF) is used. Moreover, the equilibrium solution is derived analytically for the special case of a lottery CSF, and approximately for the remaining cases. Note that the main difficulty which arises in multi-stage contests if agents are allowed to be heterogeneous is that continuation values in early stages become endogenous due to feedback effects across different branches of the game. Therefore, I analyze those feedback effects in some detail, as well as several other properties of the equilibrium solution.

Given the wide variety of potential applications, it is almost suprising that multi-stage elimination contests with a Tullock CSF have been considered mainly for the most simple case in which all participating agents are identical so far. Even a very recent paper on the optimal design of multi-stage contests by Fu & Lu (2009) entirely focusses on setting with homogenous agents, yet research on settings with heterogeneous agents is encouraged in the conclusion.[1] Only special cases of the arguably more relevant case where agents can be of different types have been analyzed in the past: In the theoretical literature on contest design, Stein & Rapoport (2004) compare the behavior of asymmetric agents in two-stage contests with different orderings of within and between group competitions. However, since homogeneity is assumed within each group, the major complication that arises in a multi-stage compe-

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[1] Another example is the paper by Gradstein & Konrad (1999), where single- and multi-stage contests are compared for the case of homogeneous agents only.

tition between heterogeneous contestants is avoided, namely endogenous continuation values in early stages of the game. Other authors focus on specific applications of multi-stage contests: For example, Rosen (1986) uses multi-stage pairwise elimination contest structure to model a promotion tournament in a firm, and presents some rather weak numerical evidence only for such settings where agents are heterogeneous;[2]. Another example is the paper by Harbaugh & Klumpp (2005), in which the same contest structure is analyzed as in this paper, but agents can be of two different types only, and the effort choice is restricted by a simplifying assumption.[3] Finally, Klumpp & Polborn (2006) consider heterogeneous contestants in a multi-stage competition, but their contest structure is somewhat different from the one that is analyzed in this paper, because they assume that the same two agents interact repeatedly within stage 1.

Somewhat more is known about the properties of multi-stage contests with heterogeneous agents in a different branch of the contest literature, which uses a perfectly discriminating CSF, the so-called "all-pay auction".[4] Moldovanu & Sela (2006) compare one-stage and multi-stage contests and explicitly allow for heterogeneity between the contestants. However, it is the paper by Groh, Moldovanu, Sela & Sunde (2010) which is most similar to my approach. Groh et al. (2010) consider the case of four heterogeneous, optimizing agents in a two-stage pairwise elimination tournaments. They derive the mixed-strategy equilibrium and determine how players should be paired, or seeded, in stage 1 to satisfy four different optimality criteria. Although the baseline situation is the same in their and in my model, the focus is very different: Groh et al. (2010) restrict their attention entirely to the effect which the allocation of player types in stage 1 ("Seeding") has on the properties of a two-stage contest with four agents. Even though I also address this point (briefly) at the end of the paper to analyze the influence of the CSF on the properties of different Seedings, the approach in my paper is broader. I also analyze comparative statics behavior, and discuss the effect of heterogeneity in multi-stage contests on the structure of optimal prizes. Further, the feedback effect across different branches of the game is considered in some detail, as well as situations with more than two stages. Apart from that, I use a different CSF that, in contrast to the all-pay auction case does not restrict the structure of prizes in any dimension, and gives an equilibrium in pure strategies, which is an advantage if one intends to test certain predictions of the model in a controlled laboratory experiment.

I proceed as follows: In section 2, the model of a multi-stage pairwise elimination contest is pre-

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[2] He determines the optimal structure of prizes under the assumption that agents are perfectly homogenous; optimality refers to constant incentives for effort provision across stages. In the last section of the paper, he discusses by use of numerical evidence in how far his results do hold in settings with heterogeneous agents.

[3] The authors assume that a fixed endowment which is without intrinsic value to any player is split across the two-stages of the contest game.

[4] In such a setting, a marginal lead in terms of contest investments implies a winning probability of 1.

sented. I start with the subgame perfect Nash-equilibrium solution for a situation with four agents who compete in two stages and analyze the effect that works across stage 1 interaction. Then, possible extensions to more general settings with three or more stages are discussed. In section 3, several properties of the simplest multi-stage contest with two stages are discussed: Some comparative static results are presented, the properties of different Seedings with respect to the criteria proposed by Groh et al. (2010) are analyzed, and finally the issue of optimal prizes is addressed briefly. Section 4 concludes.

2 Modeling multi-stage pairwise elimination contests

I will start by considering the simplest case of a multi-stage pairwise elimination contest, which is a contest with two-stages and four agents. At the end of this section, I will discuss in how far the solution technique presented works in contests with three or more stages.

In a two-stage contest, there are three pairwise interactions: Two in stage 1, and a third one in stage 2 between the two stage 1 winners. The four agents are assumed to be risk neutral and identical apart from the individual effort productivity parameter a_i which determines the player type. The higher a_i, the *stronger* is the agent.[5] Players are perfectly informed about their own type and the type of the remaining three agents participating in the contest. They do not know, however, which agent they will meet in stage 2 of the game, since decisions of all agents in stage 1 are made simultaneously.[6] Instead, they form expectations with respect to the probability that they meet a certain agent in stage 2.

Three prizes are allowed for: The prize $P^H \geq 0$ is awarded to the winner of the stage 2 subgame, while the two agents who reach stage 2 each receive $P^L \geq 0$. Consequently, the winner of stage 2 receives $P^L + P^H$, and the overall amount of resources which are used for prizes is equal to the sum $2P^L + P^H$.[7] Each pairwise interaction is modeled using a Tullock contest success function (Tullock 1980): The winning probability p_{ij} of agent i who competes with agent j is defined as

$$p_{ij} = \frac{a_i x_i^r}{a_i x_i^r + a_j x_j^r},$$

where x_i (x_j) is the effort provided by agent i (j); r is the discriminatory power of the contest success function, while the parameter a_i (a_j) measures the effort productivity of agent i (j).[8]

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[5] It is without loss of generality that I model heterogeneity between agents in terms of effort productivity. All the subsequent results do also hold if agents have different valuation or cost of effort parameters.

[6] Decisions can be made sequentially, as long as no agent is informed about the decision of any other agent before he has made his own decision.

[7] The solution to the game does not change under the assumption that only one or two prizes exist; if two prizes exist, one would be for the winner of the second stage game, and one for the loser of the second stage game.

[8] This contest success function has been axiomatized by Clark & Riis (1998*b*).

In the remainder of this section, I will first characterize a solution for arbitrary degrees of discriminatory power r. As will be shown below, an analytical closed form solution for equilibrium behavior in stage 1 cannot be derived, but equilibrium behavior in stage 1 can be determined rather precisely by using bounds which restrict the range of potential actions that are consistent with equilibrium behavior; it is also possible to show that the two-stage contest has a unique equilibrium in pure-strategies if heterogeneity and the discriminatory power are not too high. In the second part of this section, a closed form solution is presented for the special case of a lottery CSF with discriminatory power $r = 1$. Finally, I discuss in how far the previous analysis can be extended to contests with more than two stages.

2.1 A two-stage Tullock contest with discriminatory power r

It is assumed without loss of generality that agents 1 and 2 meet in one of the two interactions in stage 1, while agents 3 and 4 compete in the remaining one.[9] The equilibrium concept needed to solve this game is subgame perfect Nash: one has to solve stage 2 of the game first, or, to be precise, all possible constellations of second stage games, and subsequently consider the first stage (backwards induction), taking actions in the second stage as given. Therefore, I start by solving all potential stage 2 interactions, and subsequently consider the two pairwise interactions in stage 1.

2.1.1 Solving stage 2

Since only one agent from each subgame proceeds to the second stage, four constellations in the stage 2 subgame are possible: (i) agent 1 - agent 3, (ii) agent 1 - agent 4, (iii) agent 2 - agent 3, or (iv) agent 2 - agent 4.[10] Note that any of those games is a simple interaction between two heterogeneous agents, a situation which has been studied by Allard (1998) and Nti (1999) in slightly different settings.[11] I will now derive a solution for the general case where agent i meets agent j. It is assumed without loss of generality that agent i is stronger than agent j, i.e. the relation $a_i \geq a_j$ does hold. $x_{ij} \geq 0$ ($x_{ji} \geq 0$) denotes the effort of agent i (j).[12] The two agents compete for the prize P^H and choose their efforts in such a way as to maximize their expected payoff $\pi_i(i-j)$ and $\pi_j(j-i)$, respectively. Formally, the

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[9] This assumptions simplifies the subsequent analysis, since it limits the number of possible constellations in stage 2.

[10] Note that two or more of those possibilities can be strategically equivalent, if at least two agents are of the same type, i.e. if their effort productivities are identical.

[11] Allard (1998) focuses on existence and uniqueness of equilibria in heterogeneous Tullock contests, while Nti (1999) presents extensive comparative static results models for heterogeneous two player contests.

[12] The assumption that $x_{ij} \geq 0$ is standard in the literature, since negative investments into a contest do not make sense.

optimization problems are as follows:

$$\begin{aligned}
\max_{x_i \geq 0} \pi_i(i-j) &= \frac{a_i x_{ij}^r}{a_i x_{ij}^r + a_j x_{ji}^r} P^H - x_{ij} \\
\max_{x_j \geq 0} \pi_j(j-i) &= \frac{a_j x_{ji}^r}{a_i x_{ij}^r + a_j x_{ji}^r} P^H - x_{ji}.
\end{aligned}$$

Recall that P^H is the prize for winning stage 2, while r is the discriminatory power of the CSF; the parameter a_i measures the effort producitivity of agent i and introduces heterogeneity between agents into the model, if a_i is stricly larger than a_j.[13] Taking first derivatives gives the following system of optimality conditions:

$$r a_i a_j x_{ij}^{r-1} x_{ji}^r P^H - (a_i x_{ij}^r + a_j x_{ji}^r)^2 = 0 \tag{1}$$

$$r a_i a_j x_{ji}^{r-1} x_{ij}^r P^H - (a_i x_{ij}^r + a_j x_{ji}^r)^2 = 0 \tag{2}$$

The above conditions are necessary and sufficient for the unique pure-strategy equilibrium if the discriminatory power r is not too high.[14] Nti (1999) derived a formal condition which assures that the equilibrium is in pure strategies in two player contests with heterogeneous agents. I assume that this condition is satisfied.

Assumption 1. *The contest between agents i and j has a unique pure strategy equilibrium, i.e. the following condition holds:*

$$r \leq \frac{a_j}{a_i} + 1.$$

Under Assumption 1, the system of equations (1) and (2) fully characterizes the equilibrium, i.e. the optimality conditions are necessary and sufficient. Combination of those equation implies equilibrium efforts

$$x_{ij}^* = x_{ji}^* = r \frac{a_i a_j}{(a_i + a_j)^2} P^H. \tag{3}$$

Both agents provide the same effort, even though their productivity parameters a_i and a_j can be different. This does not hold, however, for expected equilibrium payoffs. Inserting x_{ij}^* and x_{ji}^* in the

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[13] All productivity parameters are non-negative and finite.

[14] Mixed strategies which occur in homogeneous two player Tullock contests if $r > 0$ are discussed by Baye, Kovenock & de Vries (1994).

payoff functions π_i gives the expected equilibrium payoffs

$$\pi_i^*(i-j)=\frac{a_i^2+(1-r)a_ia_j}{(a_i+a_j)^2}P^H, \qquad \text{and} \qquad \pi_j^*(j-i)=\frac{a_j^2+(1-r)a_ia_j}{(a_i+a_j)^2}P^H. \tag{4}$$

Recall that agent i is assumed to be stronger than agent j, i.e. the relation $a_i \geq a_j$ does hold. Inspection of the expected equilibrium payoffs reveals that $\pi_i^*(i-j) \geq \pi_j^*(j-j)$, i.e. the expected equilibrium payoff of the stronger agent is higher, as intuition would suggest. Further, the difference (or ratio) of the expected equilibrium payoffs is increasing in the difference (or ratio) of the effort productivity parameters.

2.1.2 Solving stage 1

Without loss of generality, it is assumed that agent 1 is stronger than agent 2, while agent 3 is stronger than agent 4, i.e. $a_1 \geq a_2$, and $a_3 \geq a_4$ do hold. Recall that there are two interactions in stage 1: One between agents 1 and 2, and another one between agents 3 and 4. I will first consider the former one. Let's assume that y_{ij} is the stage 1 effort by agent i who meets j. Then, agents 1 and 2 face the following maximization problems:

$$\begin{aligned}
\max_{y_{12}\geq 0} \Pi_1 &= \frac{a_1y_{12}^r}{a_1y_{12}^r+a_2y_{21}^r}\left[P^L+\underbrace{\frac{a_3y_{34}^r}{a_3y_{34}^r+a_4y_{43}^r}\pi_1^*(1-3)+\frac{a_4y_{43}^r}{a_3y_{34}^r+a_4y_{43}^r}\pi_1^*(1-4)}_{P_1^c(y_{34},y_{43})}\right]-y_{12}\\
\max_{y_{21}\geq 0} \Pi_2 &= \frac{a_2y_{21}^r}{a_1y_{12}^r+a_2y_{21}^r}\left[P^L+\underbrace{\frac{a_3y_{34}^r}{a_3y_{34}^r+a_4y_{43}^r}\pi_2^*(2-3)+\frac{a_4y_{43}^r}{a_3y_{34}^r+a_4y_{43}^r}\pi_2^*(2-4)}_{P_2^c(y_{34},y_{43})}\right]-y_{21}.
\end{aligned}$$

Note that the prizes for which agents compete in stage 1 consist of two parts: (1) each agent who reaches stage 2 receives P^L, and (2) each agent i who reaches stage 2 has the chance to win the prize P^H; the winning probability and the effort costs of the stage 2 interaction are included in the expected equilibrium payoffs, $\pi_i^*(i-j)$ (see above). I call this second part of the prize the *continuation value*, i.e. $P_1^c(y_{34},y_{43})$ and $P_2^c(y_{34},y_{43})$, respectively, are the continuation values of agents 1 and 2. Due to the assumption that agent 1 is stronger than agent 2, it must hold that $P_1^c(y_{34},y_{43}) \geq P_2^c(y_{34},y_{43})$, since the expected equilibrium payoff of a stage 2 participation is higher for agent 1 than for agent 2, no matter whether he meets agent 3 or 4 in stage 2.[15] Note, however, that the continuation values

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[15] See the discussion in the previous section that the ratio or difference of expected equilibrium payoffs is increasing in the ratio or difference of productivity parameters, which measure the strength of an agent.

also depend on actions of agents 3 and 4 in the other stage 1 interaction, because meeting agent 3 in stage 2 has a different value for agents 1 or 2 than meeting agent 4 as long as agents 3 and 4 are not of the same type. Therefore, in general each agent does not only play a best response to his immediate opponent, but in addition he also considers the stage 1 actions of his prospective opponent(s) in stage 2. This indirect effect which connects the two stage 1 interactions makes the continuation values endogenous and constitutes the main difficulty in solving multi-stage contests with heterogeneous agents. If the continuation values were known, the two stage 1 interactions would be independent from one another and the standard solution technique used for stage 2 could be employed. However, one can still determine the necessary equilibrium conditions which can be used to solve the model once the continuation values are determined. The maximization problems of agents 1 and 2 imply that the following two first-order optimality conditions do hold:

$$\begin{aligned} r a_1 a_2 y_{21}^r y_{12}^{r-1}[P^L + P_1^c(y_{34}, y_{43})] &= (a_1 y_{12}^r + a_2 y_{21}^r)^2 \qquad (5) \\ r a_1 a_2 y_{12}^r y_{21}^{r-1}[P^L + P_2^c(y_{34}, y_{43})] &= (a_1 y_{12}^r + a_2 y_{21}^r)^2 \qquad (6) \end{aligned}$$

Combining those equations, one can show that the ratio of equilibrium efforts is as follows in equilibrium:

$$\frac{y_{21}^*}{y_{12}^*} = \frac{a_4[P^L + \pi_2^*(2-4)]\left(\frac{y_{43}}{y_{34}}\right)^r + a_3[P^L + \pi_2^*(2-3)]}{a_4[P^L + \pi_1^*(1-4)]\left(\frac{y_{43}}{y_{34}}\right)^r + a_3[P^L + \pi_1^*(1-3)]} \equiv G\left(\frac{y_{43}}{y_{34}}\right), \qquad (7)$$

i.e. the ratio of equilibrium efforts is equal to the ratio of the prizes, a relation known from Nti (1999). Due to the assumption that agent 1 is stronger than agent 2 ($a_1 \geq a_2$), it must hold that the continuation value for agent 1 is at least as high as the one for agent 2. Consequently, it holds that $0 \leq \frac{y_{21}^*}{y_{12}^*} \leq 1$. Further, note that the ratio of equilibrium efforts $\frac{y_{21}^*}{y_{12}^*}$ is a function of the effort ratio in the second stage 1 interaction between agents 3 and 4, which implies that knowledge of equilibrium efforts y_{34}^* and y_{43}^* is, strictly speaking, not necessary to determine the contination values of agents 1 and 2; all one needs to know is the ratio of equilibrium efforts. An analogous relation can be determined for the stage 1 interaction between agents 3 and 4, namely

$$\frac{y_{43}^*}{y_{34}^*} = \frac{a_2[P^L + \pi_4^*(4-2)]\left(\frac{y_{21}}{y_{12}}\right)^r + a_1[P^L + \pi_4^*(4-1)]}{a_2[P^L + \pi_3^*(3-2)]\left(\frac{y_{21}}{y_{12}}\right)^r + a_1[P^L + \pi_3^*(3-1)]} \equiv R\left(\frac{y_{21}}{y_{12}}\right). \qquad (8)$$

Again, the ratio of equilibrium efforts $\frac{y_{43}^*}{y_{34}^*}$ is between zero and one, since agent 3 is assumed to be stronger than agent 4 ($a_3 \geq a_4$), and it is a function of the ratio of efforts in the other stage 1 interaction.

Summing up, equation (7) assures that agents 1 and 2 play mutually best responses for given continuation values. In the same manner, equation (8) guarantees that agents 3 and 4 choose effort levels which are consistent with Nash equilibrium behavior for given continuation values. Finally, the system of equations (7) and (8) jointly determines the equilibrium continuation values of the stage 1 subgame. Before I further consider the equilibrium of the stage 1 subgame, I have to make sure that, given equilibrium continuation values, the (partial) equilibrium in each of the two pairwise stage 1 interactions is in pure strategies, since the optimality conditions are not necessary and sufficient otherwise. Assumption 2 assures that this is the case.

Assumption 2. *Let* $P_1^c(y_{34}^*, y_{43}^*)$*,* $P_2^c(y_{34}^*, y_{43}^*)$*,* $P_3^c(y_{12}^*, y_{21}^*)$ *and* $P_4^c(y_{12}^*, y_{21}^*)$ *be the equilibrium continuation values. Then, the following relations are satisfied:*

$$r \leq \frac{a_2}{a_1}\left(\frac{P^L + P_2^c(y_{34}^*, y_{43}^*)}{P^L + P_1^c(y_{34}^*, y_{43}^*)}\right)^r + 1, \qquad \text{and} \qquad r \leq \frac{a_4}{a_3}\left(\frac{P^L + P_4^c(y_{12}^*, y_{21}^*)}{P^L + P_3^c(y_{12}^*, y_{21}^*)}\right)^r + 1.$$

All that is needed to determine equilibrium effort levels of all four players is the knowledge of either $G^*(\frac{y_{43}^*}{y_{34}^*}) = \frac{y_{21}^*}{y_{12}^*}$, or $R^*(\frac{y_{21}^*}{y_{12}^*}) = \frac{y_{43}^*}{y_{34}^*}$. If one of the two equilibrium ratios is known, it is straighforward to determine the missing one. Using the expressions $R^*(\frac{y_{21}^*}{y_{12}^*})$ and $G^*(\frac{y_{43}^*}{y_{34}^*})$, equilibrium continuation values are defined as follows:

$$P_1^{c*} = \frac{a_3\pi_1^*(1-3) + a_4\pi_1^*(1-4)R^*(\cdot)}{a_3 + a_4R^*(\cdot)}, \qquad P_2^{c*} = \frac{a_3\pi_2^*(2-3) + a_4\pi_2^*(2-4)R^*(\cdot)}{a_3 + a_4R^*(\cdot)} \tag{9}$$

$$P_3^{c*} = \frac{a_1\pi_3^*(3-1) + a_2\pi_3^*(3-2)G^*(\cdot)}{a_1 + a_2G^*(\cdot)}, \qquad P_4^{c*} = \frac{a_1\pi_4^*(4-1) + a_2\pi_4^*(4-2)G^*(\cdot)}{a_1 + a_2G^*(\cdot)} \tag{10}$$

If the equilibrium continuation values are known, the two stage 1 interactions are independent from one another, and each can be solved along the same lines as the stage 2 interaction in section 2.1.1. Then, equilibrium effort levels in stage 1 are:

$$y_{12}^* = \frac{a_1a_2[P^L + P_1^{c*}]}{(a_1 + a_2G^*(\cdot))^2}, \qquad y_{21}^* = \frac{a_1a_2[P^L + P_2^{c*}]}{(a_1 + a_2G^*(\cdot))^2} \tag{11}$$

$$y_{34}^* = \frac{a_3a_4[P^L + P_3^{c*}]}{(a_3 + a_4R^*(\cdot))^2}, \qquad y_{43}^* = \frac{a_3a_4[P^L + P_4^{c*}]}{(a_3 + a_4R^*(\cdot))^2}. \tag{12}$$

Knowledge of stage 1 equilibrium efforts allows for the determination of expected equilibrium payoffs, total equilibrium effort expenditures, equilibrium winning probabilities, and the like, i.e. the stage 1 subgame of the two-stage contest is fully solved.

In the above solution, I simply assumed that $G^*(\cdot)$ and $R^*(\cdot)$ are known. However, when combining equations (7) and (8) who implicitly define both $G^*(\cdot)$ and $R^*(\cdot)$, it is impossible to derive a closed form analytical expression for either of the two ratios in the general case; only two special cases can be solved: First, it is possible to solve the game if either $a_1 = a_2$ or $a_3 = a_4$, or both. If this is the case, it is easy to show that at least of the two ratios is equal to one, such that the missing one can be determined. The intuition for this result is simple: if any two players who interact in stage 1 are equally strong, it is obvious that their continuation values are identical, which implies that their stage 1 efforts are the same. The second special case is less obvious and will be discussed in some detail later in the paper: If $r = 1$, i.e. if a lottery CSF is used, the system of equations (7) and (8) can be solved analytically; no restrictions on the productivity parameters a_i are needed.

Even though a closed form solution for $G^*(\cdot)$ and $R^*(\cdot)$, and therefore for the stage 1 efforts cannot be derived for the general case, the two stage contest has a unique subgame perfect Nash equilibrium which exists under assumptions 1 and 2. This claim is proven in the next section.

2.1.3 The subgame perfect Nash equilibrium in pure-strategies

The subgame perfect Nash-equilibrium of a two-stage pairwise elimination tournament with four agents is given by the strategy profile

$$S^* = \left\{ \left(y_{12}^* \left(y_{21}, \frac{y_{43}}{y_{34}} \right), [x_{13}^*(x_{31}), x_{14}^*(x_{41})] \right) ; \ \ldots \ ; \ \ldots \ ; \left(y_{43}^* \left(y_{34}, \frac{y_{21}}{y_{12}} \right), [x_{41}^*(x_{14}), x_{42}^*(x_{24})] \right) \right\},$$

i.e. in stage 1, each agent plays a best response to the chosen outlay of his immediate opponent and the ratio of outlays of the remaining two agents in the other stage 1 interaction, as was shown in section 2.1.2. In stage 2, the best response is only with respect to the direct opponent (see section 2.1.1).

In the previous section, I simply claimed that the above equilibrium always exists, and that it is unique. This claim is repeated below and proven subsequently.

Theorem 1 (Existence and Uniqueness)**.** *Under Assumptions 1 and 2, the two stage Tullock contest with four agents and discriminatory power r has a unique subgame perfect equilibrium in pure strategies which exists for any combination of player types.*

Proof. See Appendix. □

For the proof, note that Assumptions 1 assure that the stage 2 subgame has a unique pure strategy equilibrium. Further, Assumption 2 implies that each of the two stage 1 interactions has a unique pure strategy equilibrium as well, conditional on given continuation values.[16] Consequently, what remains

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[16] This has been proven by Nti (1999) and Cornes & Hartley (2005), for example.

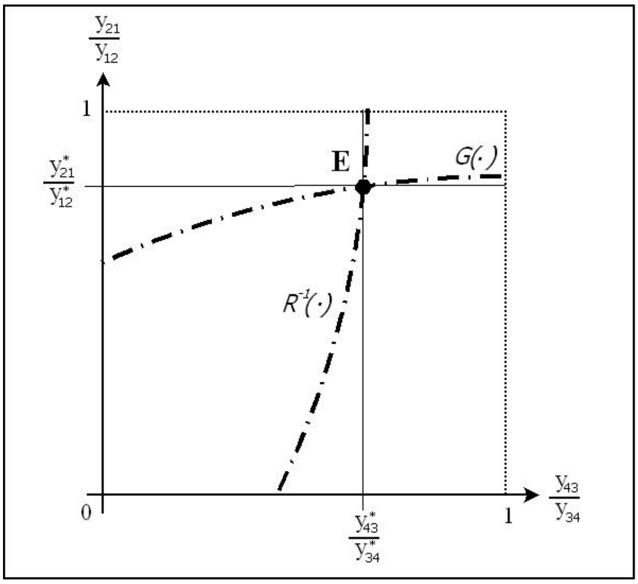

Figure 1: Intuition for the proof

to be proven is that the system of equations (7) and (8) has a solution which is unique. Figure 1 plots the two functions $G(\cdot)$ as well as $R(\cdot)$ and provides the intuition for the proof, which is that the graphs of $G(\cdot)$ and the inverse function $R^{-1}(\cdot)$ intersect exactly once in the $(\frac{y_{43}}{y_{34}}, \frac{y_{21}}{y_{12}})$-space.

Since the interaction effect across the two stage 1 interactions which is captured by the system of equations (7) and (8) is the main complication that arises in multi-stage pairwise elimination contests with heterogeneous agents, I will analyze this effect in more detail below by use of graphical illustrations which help to provide an intuition for the forces at work.

First, I will discuss which factors determine the location of the graphs. Therefore, I assume that the interaction between agents 1 and 2 is homogenous, i.e. that those two agents are equally strong. Due to this assumption, the graph for $G(\cdot)$ is a horizontal line through the point $\frac{y_{21}}{y_{12}} = 1$, as shown in the left panel of Figure 2. Then, $R^{-1}(\cdot)$ is ploted for three different degrees of heterogeneity between agents 3 and 4: high heterogeneity, which gives the equilibrium E_1, medium heterogeneity (equilibrium E_2), and homogeneity between the two agents (E_3). The higher the heterogeneity, the further to the left is the graph.[17] Recall from the analysis of the stage 2 interaction in section 2.1.1 that the efforts of two players who differ only in terms of their productivities are equal, independent of the degree of heterogeneity. Consequently, the distance of $R^{-1}(\cdot)$ fromt the vertical line through E_3 measures the additional heterogeneity between agents 3 and 4 that is caused by different valuations of a participation in stage 2. Said differently, the fact that agents 3 and 4 are (still) of different strength in later stages further increases the degree of heterogeneity in early stages of the game through the continuation values. The location of the graph for $G(\cdot)$ is affected by the same forces: It is equal to the horizontal

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[17] Note that $R^{-1}(\cdot)$ is an inverse correspondence rather than an inverse function in this example.

line through $\frac{y_{21}}{y_{12}} = 1$ that is ploted in the Figure if agents 1 and 2 are equally strong. When the heterogeneity between those two agents becomes higher, $G(\cdot)$ is shifted downwards, as continuation values become more and more different, such that de facto heterogeneity in stage 1 is increased above the heterogeneity level which is due to different productivities in the pairwise stage 1 interaction.

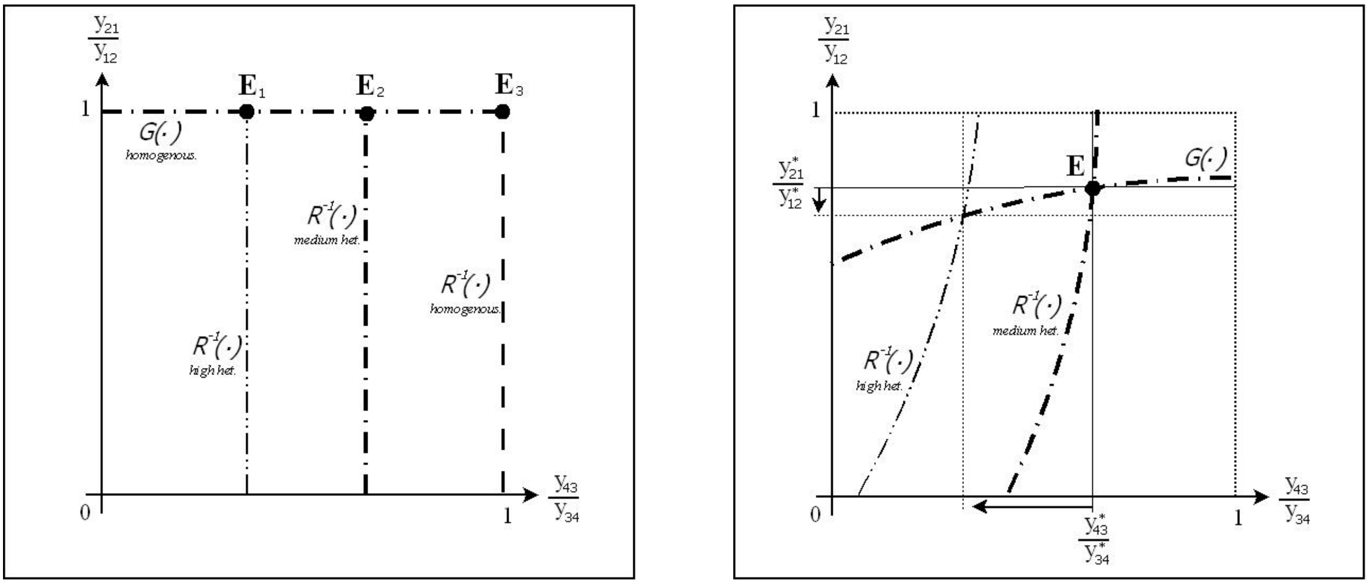

Figure 2: The effect which works across the two stage 1 interactions

The right panel of Figure 2 depicts a situation where both stage 1 interactions are heterogeneous. In contrast to the left panel of the figure which was previously discussed, there is a mutual interdependence across stage 1 interactions, since both $G(\cdot)$ and $R^{-1}(\cdot)$ are functions with a stricly positive slope. The slopes are strictly positive if and only if the two relations $a_1 > a_2$ and $a_3 > a_4$ are both satisfied. It is important to stress that there is no interdependence if (at least) one of the two stage 1 interactions is homogenous, as the left panel of the figure clearly shows.[18] Now, assume that the initial equilibrium is given by the point E. Then, heterogeneity between agents 3 and 4 is increased such that player 3 becomes stronger. As a consequence, the $R^{-1}(\cdot)$ function shifts to the left, and ceteris paribus, the effort of agent 4 decreases relative to the effort of agent 3. Due to interaction effects between the two stage 1 interactions, this change affects the behavior of agents 1 and 2, even though their types remain unchanged. There are two different channels: *First*, the intersection of the two functions is further to the left, and since $G(\cdot)$ is increasing, the equilibrium ratio of efforts in the second stage 1 interaction decreases, i.e. the weaker agent 2 reduces his effort provision relative to the effort of agent 1. The strength of this effect depends on the slope of $G(\cdot)$. Usually, $G(\cdot)$ is much flatter than depicted in the plot, such that the change in the ratio $\frac{y_{21}^*}{y_{12}^*}$ due to a stronger agent 3 is rather small. *Second*, the

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[18] The ratio $\frac{y_{21}}{y_{12}}$ is equal to one, independent of $\frac{y_{43}}{y_{34}} = 1$.

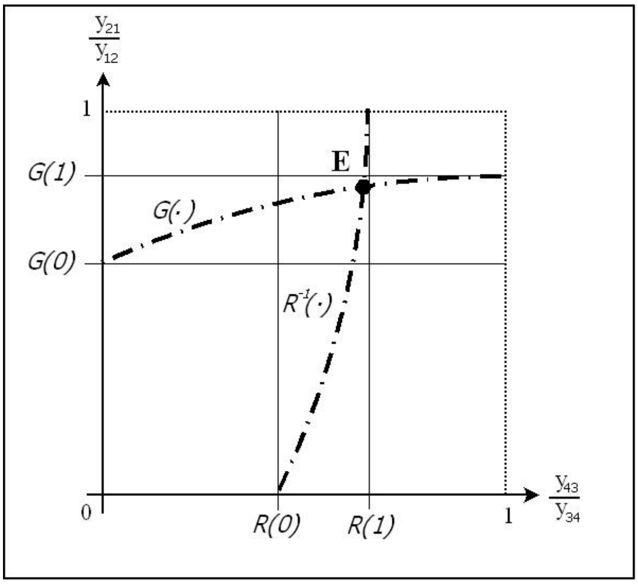

Figure 3: Bounds for the equilibrium ratios

$G(\cdot)$ shifts downwards and becomes somewhat steeper, which further reduces the equilibrium ratio $\frac{y^*_{21}}{y^*_{12}}$. Intuitively, the fact that agent 3 is stronger than before has a more pronounced negative effect on the continuation value of agent 2 than on the continuation value of agent 1, especially for low values of $\frac{y^*_{43}}{y^*_{34}}$ when the probability to meet the strong agent 3 is high. Note that this second effect is not depicted in the figure for clarity reasons.

Summing up, the effect of mutual interdependence in stage 1 does only exist if both stage 1 interactions are heterogeneous. This explains why a closed form solution for the general model can be derived for all cases but this one. If there is mutual interdependence, the strength of the effect depends on the slope of the functions: The steeper $G(\cdot)$ and the flatter $R^{-1}(\cdot)$, the more important is the interdependence effect. One can show formally that the steepness of $G(\cdot)$, which is the equilibrium ratio function for the interaction between agents 1 and 2, is increasing in the degree of heterogeneity between agents 3 and 4. An analogous finding applies to $R^{-1}(\cdot)$.

Finally, Figure 3 illustrates that there is a straightforward way to approximate the solution if we cannot determine it analytically. The key insight is that the stage 1 equilibrium effort ratios are bounded. Note that both $\frac{y^*_{43}}{y^*_{34}} \in [R(0), R(1)]$ and $\frac{y^*_{21}}{y^*_{12}} \in [G(0), G(1)]$ must hold. Therefore, one can bound the two equilibrium ratios as follows:

$$\frac{y^*_{21}}{y^*_{12}} \in [G(R(0)), G(R(1))] \qquad \text{and} \qquad \frac{y^*_{43}}{y^*_{34}} \in [R(G(0)), R(G(1))].$$

For reasonable degree of heterogeneity between agents, those bounds are very narrow, such that the equilibrium solution can be approximated with a high precision. The upper bound is a good approximation if heterogeneity between agents 1 and 2 (or agents 3 and 4, respectively), is low, since, intuitively, the continuation values of the two agents do not differ much then. Contrary, the lower bound is preferable if heterogeneity is high due to the same reasoning. However, the approximation becomes most precise if one uses $\frac{y_{21}}{y_{12}} = \frac{a_2}{a_1+a_2}$ as well as $\frac{y_{43}}{y_{34}} = \frac{a_4}{a_3+a_4}$ rather than one or zero, respectively. Obviously, both expressions are between zero and one, and they account for the degree of heterogeneity in each stage 1 interaction. Extensive numerical evidence suggests that the percentage deviation of this approximated solution from the correct one is below 1 percentage point for arbitrary degrees of heterogeneity. Using this approximation technique, the equilibrium effort ratios satisfy

$$\frac{y^*_{21}}{y^*_{12}} \approx G\left[R\left(\frac{a_2}{a_1+a_2}\right)\right] \qquad \text{and} \qquad \frac{y^*_{43}}{y^*_{34}} \approx R\left[G\left(\frac{a_4}{a_3+a_4}\right)\right].$$

2.2 A two-stage Tullock contest for the lottery CSF ($r = 1$)

The approximation technique presented previously is not necessary if a lottery CSF with discriminatory power $r = 1$ is used. For this special case, a closed-form analytical solution can be derived. Imposing the assumption $r = 1$ on equations (7) and (8), one obtains the following system of equations:

$$\begin{aligned} \frac{y^*_{21}}{y^*_{12}} &= \frac{a_4[P^L + \pi^*_2(2-4)]\frac{y_{43}}{y_{34}} + a_3[P^L + \pi^*_2(2-3)]}{a_4[P^L + \pi^*_1(1-4)]\frac{y_{43}}{y_{34}} + a_3[P^L + \pi^*_1(1-3)]} \equiv G\left(\frac{y_{43}}{y_{34}}\right) & (13) \\ \frac{y^*_{43}}{y^*_{34}} &= \frac{a_2[P^L + \pi^*_4(4-2)]\frac{y_{21}}{y_{12}} + a_1[P^L + \pi^*_4(4-1)]}{a_2[P^L + \pi^*_3(3-2)]\frac{y_{21}}{y_{12}} + a_1[P^L + \pi^*_3(3-1)]} \equiv R\left(\frac{y_{21}}{y_{12}}\right). & (14) \end{aligned}$$

To make the subsequent analysis tractable, I define the functions κ, ϕ, λ, μ, θ, γ, ψ, and ζ, which depend on exogenous heterogeneity and prize parameters only.[19] Then, inserting $R(\frac{y_{21}}{y_{12}})$ in (13) and

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[19] It holds that:

$$\begin{aligned} \kappa &= a_3(P^L + \pi^*_1(1-3))(P^L + \pi^*_3(3-1)) + a_4(P^L + \pi^*_1(1-4))(P^L + \pi^*_4(4-1)) \\ \phi &= a_3(P^L + \pi^*_1(1-3))(P^L + \pi^*_3(3-2)) + a_4(P^L + \pi^*_1(1-4))(P^L + \pi^*_4(4-2)) \\ \lambda &= a_3(P^L + \pi^*_2(2-3))(P^L + \pi^*_3(3-1)) + a_4(P^L + \pi^*_2(2-4))(P^L + \pi^*_4(4-1)) \\ \mu &= a_3(P^L + \pi^*_2(2-3))(P^L + \pi^*_3(3-2)) + a_4(P^L + \pi^*_2(2-4))(P^L + \pi^*_4(4-2)) \\ \theta &= a_1(P^L + \pi^*_3(3-1))(P^L + \pi^*_1(1-3)) + a_2(P^L + \pi^*_3(3-2))(P^L + \pi^*_2(2-3)) \\ \gamma &= a_1(P^L + \pi^*_3(3-1))(P^L + \pi^*_1(1-4)) + a_2(P^L + \pi^*_3(3-2))(P^L + \pi^*_2(2-4)) \\ \psi &= a_1(P^L + \pi^*_4(4-1))(P^L + \pi^*_1(1-3)) + a_2(P^L + \pi^*_4(4-2))(P^L + \pi^*_2(2-3)) \\ \zeta &= a_1(P^L + \pi^*_4(4-1))(P^L + \pi^*_1(1-4)) + a_2(P^L + \pi^*_4(4-2))(P^L + \pi^*_2(2-4)). \end{aligned}$$

$G(\frac{y_{43}}{y_{34}})$ in (14), respectively, results in

$$\frac{y^*_{21}}{y^*_{12}} = \frac{a_2\mu\frac{y_{21}}{y_{12}} + a_1\lambda}{a_2\phi\frac{y_{21}}{y^*_{12}} + a_1\kappa} \tag{15}$$

$$\frac{y^*_{43}}{y^*_{34}} = \frac{a_4\zeta\frac{y_{43}}{y_{34}} + a_3\psi}{a_4\gamma\frac{y_{43}}{y^*_{34}} + a_3\theta}. \tag{16}$$

In equilibrium, it must obviously hold that $\frac{y_{21}}{y_{12}} = \frac{y^*_{21}}{y^*_{12}}$, as well as $\frac{y_{43}}{y_{34}} = \frac{y^*_{43}}{y^*_{34}}$. Imposing this and rearranging gives two quadratic equations

$$\left(\frac{y^*_{21}}{y^*_{12}}\right)^2 + \frac{a_1\kappa - a_2\mu}{a_2\phi}\left(\frac{y^*_{21}}{y^*_{12}}\right) - \frac{a_1\lambda}{a_2\phi} = 0 \qquad \text{and} \qquad \left(\frac{y^*_{43}}{y^*_{34}}\right)^2 + \frac{a_3\theta - a_4\zeta}{a_4\gamma}\left(\frac{y^*_{43}}{y^*_{34}}\right) - \frac{a_3\psi}{a_4\gamma} = 0,$$

respectively, which are independent from one another. Each of those equations can be solved analytically, which gives the equilibrium ratios $\frac{y^*_{21}}{y^*_{12}}$ as well as $\frac{y^*_{43}}{y^*_{34}}$. The ratios satisfy:[20]

$$\frac{y^*_{21}}{y^*_{12}} = \frac{a_2\mu - a_1\kappa + \sqrt{[a_2\mu - a_1\kappa]^2 + 4a_1a_2\phi\lambda}}{2a_2\phi} = G^*\left(\frac{y^*_{43}}{y^*_{34}}\right) \tag{17}$$

$$\frac{y^*_{43}}{y^*_{34}} = \frac{a_4\zeta - a_3\theta + \sqrt{[a_4\zeta - a_3\theta]^2 + 4a_3a_4\psi\gamma}}{2a_4\gamma} = R^*\left(\frac{y^*_{21}}{y^*_{12}}\right). \tag{18}$$

Even though the bounds which allow for an approximated solution are not needed for this special case, I will still determine them to compare the analytical solution with the approximated one. From (15) and (16), it follows that the equilibrium ratios of effort are bounded as follows if $r = 1$:

$$\frac{y^*_{21}}{y^*_{12}} \in \left[\frac{\lambda}{\kappa}, \frac{a_2\mu + a_1\lambda}{a_2\phi + a_1\kappa}\right] \qquad \text{and} \qquad \frac{y^*_{43}}{y^*_{34}} \in \left[\frac{\psi}{\theta}, \frac{a_4\zeta + a_3\psi}{a_4\gamma + a_3\theta}\right].$$

Using the ratios $\frac{y_{21}}{y_{12}} = \frac{a_2}{a_1+a_2}$ as well as $\frac{y_{43}}{y_{34}} = \frac{a_4}{a_3+a_4}$ as suggested in the previous section, I get the expressions

$$\frac{y^*_{21}}{y^*_{12}} \approx \frac{a_2^2\mu + (a_1^2 + a_1a_2)\lambda}{a_2^2\phi + (a_1^2 + a_1a_2)\kappa} \qquad \text{and} \qquad \frac{y^*_{43}}{y^*_{34}} \approx \frac{a_4^2\zeta + (a_3^2 + a_3a_4)\psi}{a_4^2\gamma + (a_3^2 + a_3a_4)\theta}$$

for the approximated equilibrium effort ratios. Comparing them with the analytical solution in (17) and (18), one can show that they depend on the same parameters, and have qualitatively identical comparative statics properties. Numerical evidence shows that the approximated solution is always marginally higher than the "correct" analytical one.

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[20] Note that the quadratic equation has two roots, only one of which is positive. The negative root is irrelevant for the question at hand and therefore dropped.

2.3 More than two stages

So far, I considered only two-stage pairwise elimination contests with four agents. In this section, the additional complications which arise in contests with more than two stages are discussed.

Figure 4 illustrates the case of a three-stage contest with eight agents. Now, there are three instead of two subgames. As for the two-stage contest, one has to solve the game via backwards induction, i.e. I start with subgame 3, continue with subgame 2, and finally consider subgame 1.

Subgame 3 has exactly the same structure as the pairwise interaction discussed in section 2.1.1. The only difference is that sixteen different pairings are possible in the last stage of the three-stage contest, as compared to only four in the two-stage version of the model. Recall that each of those interactions can be solved analytically for any degree of heterogeneity and any discriminatory power as long as Assumption 1 is satisfied. The structure of subgame 2 is identical to the one of stage 1 in the two-stage contest, as discussed in section 2.1.2. However, now there are four possible combinations of agents each of the two interaction E and F, which gives *sixteen* different situations in subgame 2, namely $\{(1-3),(5-7)\};\{(1-3),(5-8)\};...$ and so on. Solving all of them, either analytically if $r=1$, or approximately otherwise, is tedious, but without any problems that were not discussed previously.

A new complication arises in subgame 1, where each interaction depends on all three remaining interactions of the subgame, i.e. agents 1 and 2 in pairing A play a best response to each other, *and* to the ratio of efforts by agents 3 and 4, by 5 and 6, and by 7 and 8. The best way to illustrate this point is to analyze the structure of the continuation values. I denote the effort of agent i in stage 1 by z_i and consider the continuation value P_1 of agent 1 in subgame 3:

$$\begin{aligned} P_1\left(\frac{z_3}{z_4},\frac{z_5}{z_6},\frac{z_7}{z_8}\right) &= p_{34}(p_{56}[p_{78}\times S_{1357}+p_{87}\times S_{1358}]+p_{65}[p_{78}\times S^{1367}+p_{87}\times S^{1368}]) \\ &+ p_{43}(p_{56}[p_{78}\times S_{1457}+p_{87}\times S_{1458}]+p_{65}[p_{78}\times S_{1467}+p_{87}\times S_{1468}]) \end{aligned}$$

S_{1ijk} is the expected payoff which a particular constellation of agents in subgame 2 has for agent 1, e.g. meeting agent 3 in interaction E while agents 5 and 7 compete in F is of value S_{1357}; p_{ij} is the probability that agent i wins against agent j, and it is a function of the ratio $\frac{z_i}{z_j}$. Therefore, as in the case with four agents, probabilities make the continuation values endogenous; now, however, the continuation value depends on three endogenously determined probabilities rather than only one. The first step is to consider the right (1,2,3,4) and left (5,6,7,8) branches separately, i.e., solve the interdependence between interactions A and B as well as between C and D. Each of the two branches is of the same structure as stage 1 in a two-stage contest (see discussion in 2.1.2). Then, however, there

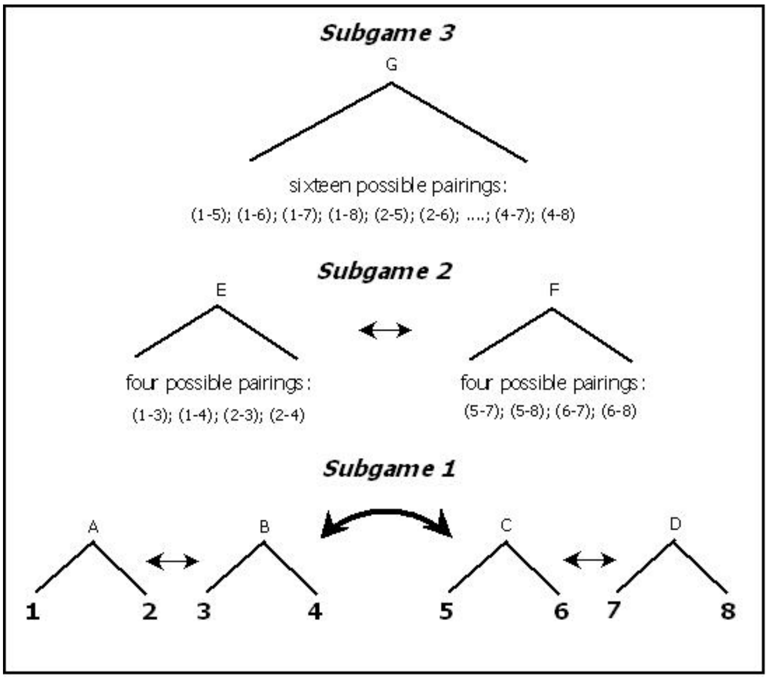

Figure 4: The eight player case

is still an interdependence across the two branches. Formally, the problem can be described as follows:

$$\frac{z_2^*}{z_1^*}/\frac{z_4^*}{z_3^*} = \frac{P_2\left(\frac{z_6}{z_5},\frac{z_8}{z_7}\right) \times P_3\left(\frac{z_6}{z_5},\frac{z_8}{z_7}\right)}{P_1\left(\frac{z_6}{z_5},\frac{z_8}{z_7}\right) \times P_4\left(\frac{z_6}{z_5},\frac{z_8}{z_7}\right)} \equiv H\left(\frac{z_6}{z_5}/\frac{z_8}{z_7}\right) \tag{19}$$

$$\frac{z_6^*}{z_5^*}/\frac{z_8^*}{z_7^*} = \frac{P_6\left(\frac{z_2}{z_1},\frac{z_4}{z_3}\right) \times P_7\left(\frac{z_2}{z_1},\frac{z_4}{z_3}\right)}{P_5\left(\frac{z_2}{z_1},\frac{z_4}{z_3}\right) \times P_8\left(\frac{z_2}{z_1},\frac{z_4}{z_3}\right)} \equiv Q\left(\frac{z_2}{z_1}/\frac{z_4}{z_3}\right). \tag{20}$$

This system of equations implicitly defines a unique solution to subgame 1, which, however, cannot be determined analytically in general, not even for $r = 1$.[21] Only special cases can be solved, where either interactions A and B, or interactions C and D are homogenous, or both. In each of those cases, there is no interaction effect across the two branches AB and CD in subgame 3. In all other cases, the solution can only be approximated, either numerically if the productivities for the eight agents are known, or analytically, using the technique introduced at the end of section 2.2. In the latter case, one gets

$$\frac{z_2^*}{z_1^*}/\frac{z_4^*}{z_3^*} \approx H\left[Q\left(\frac{a_2}{a_1+a_2}/\frac{a_4}{a_3+a_4}\right)\right] \qquad \text{and} \qquad \frac{z_6^*}{z_5^*}/\frac{z_8^*}{z_7^*} \approx Q\left[H\left(\frac{a_6}{a_5+a_6}/\frac{a_8}{a_7+a_8}\right)\right].$$

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[21] The proof for this claim includes the same steps as the one for Theorem 1. The expressions do become more complicated, however, since there are eight parameters a_i, $i \in [1, 2, ..., 8]$ to consider now rather than only four. Therefore, the formal proof is omitted.

With each additional step that is added to the multi-stage pairwise elimination contest, the number of potential pairings in later stages of the game increases. It is more problematic, however, that a new interaction effect across the two different branches of the overall game comes up in the first stage of the game for each additional stage. This complication can only be avoided if all interactions in (at least) one of the two branches are homogenous, such that the interdependence disappears. A solution for the most general multi-stage pairwise elimination contest with N players of N different player types, however, cannot be determined, and even if it were known, the resulting expressions would be too complicated to analyze theoretically. Therefore, I focus on the properties of the most simple multi-stage contest, i.e. on a two-stage contest, in the subsequent discussion section.

3 Discussion

If agents are heterogeneous, the multi-stage pairwise elimination contest has several properties which have not received much attention in the contest literature so far. In this section, I analyze comparative static behavior, properties of different player arrangements in stage 1 ("seedings" in short), and the optimal prize structure.[22] None of those topics is exhausted by the discussion in this section; rather, the main idea is to encourage future research in those dimensions, now that a solution for multi-stage pairwise elimination tournaments with heterogeneous agents is available.

3.1 Comparative Statics

In this section, I will discuss how ceteris paribus variations in the productivity parameter of an agent affect himself and the other agents participating in the contest. However, working with either the (approximated or analytical) equilibrium solution in stage 1 is extremely hard, because the resulting expressions are very complicated. Therefore, I do not consider the complete equilibrium reaction. Instead, I make the following simplifying assumption: If the productivity of agent 1 or 2 who compete in one of the two stage 1 interactions changes, the equilibrium efforts of agents 3 and 4 remain unchanged, even though their continuation values are allowed to change. In effect, I suppress the equilibrium response of agents 3 and 4 and its effect on behavior of agents 1 and 2, which is, however, usually extremely weak due to the reasons discussed in section 2.1.3. In this sense what follows is, strictly speaking, not a comparative static analysis, but rather an approximation, which is supported by extensive numerical evidence. Finally, note that I only analyze stage 1, which is, however, without

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[22] Seedings have been considered previously by Groh et al. (2010) for a perfectly discriminating all-pay auction; Rosen (1986) discussed the optimal prize structure of two-stage contests with heterogeneous agents, using numerical evidence.

loss of generality, since the continuation values account for any change in stage 2 equilibrium behavior.

I start by considering the effect which the change on an agent's productivity has within one of the two stage 1 interactions. It holds that:

Proposition 1. *The stage 1 winning probability p_{ij} of agent i who meets j in a stage 1 interaction is*

(a) increasing in the own productivity parameter a_i.

(b) decreasing in the productivity parameter a_j of the stage 1 opponent.

The same holds for the expected equilibrium payoff Π_i.

Proof. See Appendix. □

Quite intuitively, an agent benefits both in terms of his winning probability and the expected payoff if he becomes stronger: First, his chances of winning stage 1 are higher, and second, conditional on winning stage 1, he fares better in stage 2 as well, independent of his opponent in stage 2. Therefore, the continuation value of this agent increases, which further improves his chances in stage 1. If, however, the stage 1 opponent becomes stronger, the opposite holds.

Next, the effect which productivity changes in one stage 1 interaction have on the other stage 1 interaction is analyzed. In this case, it matters whether the productivity of the stronger or of the weaker agent is changed. I consider changes in the productivity of the stronger player first:

Proposition 2. *Increasing the productivity parameter of the* ***stronger*** *player in either of the two stage 1 interactions reduces the overall winning probabilities and the expected payoffs of those agents who compete in the other stage 1 interaction.*

Proof. See Appendix. □

Assuming that agents 1 and 2 compete in one, while agents 3 and 4 compete in the second stage 1 interactions, agents 1 and 2 are potential opponents in stage 2 for agents 3 and 4, and vice versa. If the stronger of the two potential stage 2 opponents becomes even stronger, the continuation values of the agents in the other stage 1 interaction must fall for two reasons: First, if the stronger agent is met, the probability to win stage 2 and the expected equilibrium payoff are lower than before. Second, the probability to meet the strong rather than the weak agent in stage 2 increases due to Proposition 1. As a consequence, the overall winning probability, which is the product of the winning probability in stage 2 and the winning probability in stage 2, for the agents in the other stage 1 interaction decreases, and the expected equilibrium payoff falls. Things may be different if the weaker of the two agents in the other stage 1 interaction becomes stronger:

Proposition 3. *Increasing the productivity parameter of the* ***weaker*** *player in any of the two stage 1 interactions may increase or decrease the winning probabilities and the expected payoffs of those agents who compete in the other stage 1 interaction.*

Proof. See Appendix. □

As in case of Proposition 2, the option values of the agents competing in the other stage 1 interaction are affected. However, there are two effects now which work in opposite directions, such that the direction of the overall effect is ambiguous. First, if the now stronger agent is met in stage 2, both the winning probability and the expected payoff are reduced, which tends to reduce the continuation value. However, at the same time, the probability to meet the weaker agent increases due to Proposition 1, and meeting the weaker agent in stage 2 is still better than meeting the stronger one for any of the two agents in the other stage 1 interaction. Which effect dominates depends on the specific values for the productivity parameters a_1, a_2, a_3, and a_4. However, it is easy to contruct situations where it is beneficial for agents in the other stage 1 interaction if the weak agent becomes stronger.

Without the intuition provided above, this result is surprising. It implies, for example, that in a situation with $a_1 \geq a_2 \geq a_3 \geq a_4$, where agents 1 and 2 compete in one, while agents 3 and 4 compete in the second stage 1 interactions, it is not unlikely that the weakest agent 4 is better of if the already second strongest agent 2 becomes even stronger. Similar results are impossible in any contest structure where all agents meet one another, either simultaneously in a one-shot contest, or sequentially in a round-robin tournament.

3.2 Seedings

For the rest of this section, I assume without loss of generality that agents are ordered by their strength, i.e. the relation $a_1 \geq a_2 \geq a_3 \geq a_4$ does hold. Note that there are three different ways to seed the four agents in stage 1 of the two-stage contest: (1-4, 2-3), where agents 1 and 2 to compete in one of the stage 1 interactions, whereas agents 3 and 4 participate in the remaining one; (1-3, 2-4), where agents 1 and 3 as well as agents 2 and 4 compete with each other in stage 1, and finally (1-2, 3-4), the setting which I considered when solving the model in section 2.1. Those three settings, or "Seedings", have different properties, i.e. even if the types of the four agents are left unchanged, seeding them differently changes the properties of the game.

The properties of different seedings have been analyzed previously by various authors, however, usually only for the case of four players, because the number of possible seedings explodes with the number of players: With 2^N players, there are $\frac{(2^N)!}{2^{(2^N-1)}}$ different seedings, i.e. 3 seedings for 4 players (as seen above), 315 seedings for 8 players, etc. Cases with up to eight players have only been addressed by the statistical literature where agents are not optimizing but probabilities are instead given exogenously; even then the problem is hard to handle analytically.[23] To my knowledge, the paper by Groh et al. (2010) is the only existing one where winning probabilities of a multi-stage pairwise elimination

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[23] See for example Schwenk (2000), Hwang (1982) or Horen & Riezman (1985).

tournament are endogenously determined by optimizing agents, as in my model. The main difference between my approach and their model is that Groh et al. (2010) use a perfectly discriminating all-pay contest success function, rather than an imperfectly discriminating one as I do.[24] Therefore, I will investigate in what follows whether, and if so, in how far, properties of different seedings are influenced by the choice of the contest success function.

Groh et al. (2010) compare the performance of the three Seedings with respect to four optimality, or fairness criteria: (1) maximization of total effort, (2) maximization of the probability of a stage 2 interaction between the two strongest agents, (3) maximization of the winning probability of the strongest agent, and (4) winning probabilities of agents are ordered according to the agent's strength. Criteria (1) to (3) are optimality criteria that are standard in the literature on Seedings, but not only there: It is a standard goal of tournament designing principals in the personnel economics literature to maximize the effort provision of his workers. Further, in case of promotion tournaments, it might be in his interest to select the most able workers or agents. A similar reasoning applies to designers of tournaments in sports. Criterium (4), however, is rather a fairness then an optimality criterium: Nobody should have a strategic disadvantage that is so high that the order of ability and winning probability is changed.

Following Groh et al. (2010), I use the capital letters A, B, and C to distinguish the three seedings and define *Seeding* A: 1-4, 2-3, *Seeding* B: 1-3, 2-4, and *Seeding* C: 1-2, 3-4. Those authors find for their specification, and under the assumption $a_1 \geq a_2 \geq a_3 \geq a_4$, that Seeding A: 1-4, 2-3 satisfies criteria (3) and (4), while Seeding B: 1-3, 2-4 fulfills both (1) and (2); none of the four criteria applies to Seeding C: 1-2, 3-4. If the prize structure in my model with an imperfectly discriminating CSF is specified in the same manner as in Groh et al. (2010), I get very similar results, with one notable exception: It still holds that Seeding A: 1-4, 2-3 satisfies criteria (3) and (4), while Seeding B: 1-3, 2-4 fulfills (2). However, criterion (1) may now be satisfied either by Seeding B: 1-3, 2-4, *or* by Seeding C: 1-2, 3-4, depending on the parameters a_1, a_2, a_3, and a_4. This finding is summarized in the subsequent Proposition:

Proposition 4. *None of the three Seedings maximizes total expected effort provision for all specifications of heterogeneity* a_1, a_2, a_3 *and* a_4.

Proof. See Appendix. □

Numerical evidence suggests that Seeding C: 1-2, 3-4 is optimal with respect to criterion (1) if the difference in strengths between the two weaker and the two stronger agents is not too big; if this difference is extreme, Seeding B: 1-3, 2-4 maximizes total expected effort provision. This makes sense intuitively: If the difference in strength between the two stronger agents and the two weaker ones is

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[24] Apart from that, there is one additional restriction on the prize structure in their model which I do not need.

relatively small, the three pairwise interactions in Seeding C: 1-2, 3-4 are relatively close, such that effort provision is high. If, however, agents 1 and 2 are much stronger than agents 3 and 4, the stage 2 interaction, where most of the effort is provided, is extremely unequal, such that the effort is very low. Then, Seeding B: 1-3, 2-4 ensures a higher effort provision due to the fact that a final between the two strongest agents is extremely likely, a situation in which effort provision is high.

Summing up, it seems that measures which depend on relative effort choices such as probabilities are not affected by the technology of the contest success function, i.e. it seems that the choice of a certain Seeding automatically implies that criteria which depend on relative effort are satisfied. [25] Yet, this does not hold for absolute measures like total expected effort provision. This implies that any contest designer who is interested in the maximization of total effort provision needs information about the importance of productivity differences for contest outcomes, i.e. on the discriminatory power of the contest success function.

3.3 Optimal Prizes

The optimality of the prize structure usually refers to the maximization of total effort expenditures.[26] It is well known in the contest literature that a unique prize maximizes total effort expenditures in one-shot contests if agents are homogenous, while multiple prizes can be optimal in heterogeneous settings. This finding holds for all major contest success functions.[27] However, little is known about the optimal prize structure in multi-stage tournaments, and even less if the agents are heterogeneous.

It is straightforward to show that a unique prize maximizes total effort expenditures in multi-stage pairwise elimination tournaments with a Tullock CSF if all participating agents are homogenous; this holds for any discriminatory power r that still allows for an equilibrium in pure strategies.[28] This result, however, may no longer hold if agents are heterogeneous. To be more precise, multiple prizes can be optimal if the interaction in stage 2 is likely to be a pairing between two agents whose strength differs a lot, while at least one of the two stage 1 interactions is fairly homogenous. An example is a situation with one superstar and three rather weak agents. In such a setting, it can be optimal to have two identical prizes, such that no effort is provided in stage 2 and the two stage 1 interactions become simple static one-shot interactions. The reason is that effort provision in the homogeneous stage 1 interaction is independent of heterogeneity, and therefore constant, while effort provision in the two-stage contest with a unique prizes goes towards zero for high degrees of heterogeneity.

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[25] See also the discussion in the conclusion of Groh et al. (2010) which relates to the paper by Horen & Riezman (1985).

[26] An exception is Rosen (1986), who considers a prize structure as optimal if effort provision is constant in all stages of a multi-stage pairwise elimination tournament.

[27] See, for example, Clark & Riis (1996), Clark & Riis (1998*a*), Clark & Riis (1998*c*), Krishna & Morgan (1998), or Moldovanu & Sela (2001). An excellent survey is provided by Sisak (2009).

[28] If $r > 2$, the equilibrium is in mixed strategies. See the respective discussion in Baye et al. (1994).

Overall, extensive numerical evidence suggests that two situations can be distinguished in tournaments with multiple stages: Either a unique prize is optimal, or it is optimal to have two identical prizes such that stage 2 is dropped and only two pairwise interactions in stage 1 remain. Essentially, this implies that the structure of prizes can be used to change the structure of the contest.

4 Conclusion

This paper presents a solution technique for multi-stage pairwise elimination contests for the case of heterogeneous contestants that differ with respect to their effort effectiveness. Elimination contests have received much attention in the contest literature, since they well describes many real life situations, such as promotion and sport tournaments, for example. So far, however, attention has focussed on cases where all agents are homogenous. I discuss the arguably more relevant general case where agents can be of different player types. Using a Tullock contest success function with discriminatory power r, it is shown under which conditions a subgame perfect Nash equilibrium exists, and how it can be determined either approximately for $r \neq 1$, or analytically for the lottery CSF with $r = 1$. So far, a solution to multi-stage pairwise elimination contests is available only for the perfectly discriminating all-pay auction contest success function (Groh et al. 2010). Most contests in reality are, however, imperfectly discriminating. Apart from that, my approach has the advantage that no restrictions on the structure of prizes are needed, and that the equilibrium is in pure strategies, which, for example, facilitates experimental testing of properties which are predicted by theory.

The main complication that arises in a multi-stage pairwise elimination contest once agents are allowed to be heterogeneneous is that continuation values in early stages become endogenous due to feedback effects across different branches of the game. I analyzed those effects in some detail for the most simple multi-stage contest with only two stages. Subsequently, additional complications that arise in more complicated settings with three stages or more were briefly discussed.

A rather short analysis of certain properties of multi-stage pairwise elimination contest with heterogeneous agents suggests that this contest format has several features that disctinctly differ from those in other contest formats. For example, I show that it can be beneficial for the weakest agent in the contest if some of the other agents become even stronger than they already are. Or, with respect to the structure of prizes, it seems that a runner-up prize for the loser in stage 2 that is smaller than the main prize is never optimal; either a unique prize or two equal prizes are optimal with respect to effort maximization. Those issues certainly deserve more attention in future research. Apart from that, it might be interesting to compare the results of the model presented in this paper to those of a model in which agents are budget-constrained. Many authors argue that agents face budget constraints in real life (Parco, Rapoport & Amaldoss 2005, Stein & Rapoport 2005, Amegashie, Cadsby & Song 2007), and to my knowledge, the implications of those constraints on behavior of agents in setups with heterogeneous agents and imperfectly discriminating CSFs have not been explored yet. This would also help to clarify the robustness of the results by Harbaugh & Klumpp (2005), who consider a special case of the model which is analyzed in this paper and assume that the endowment is of no intrinsic value to simplify their analysis.

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Appendix

Proof of Theorem 1:

First, note that Assumptions 1 assure that the stage 2 subgame has a unique pure strategy equilibrium. Further, Assumption 2 implies that each of the two stage 1 interactions has a unique pure strategy equilibrium, conditional on given continuation values.[29] Consequently, what remains to be proven is that the system of equations (7) and (8) has a solution which is unique.

For the proof, I will first show that the functions $G(\cdot)$ and $R(\cdot)$ are either strictly monotonic or constant and equal to one. If at least one of the two functions is constant, there is no interdependence between equations (7) and (8), and it is straightforward to show that there is a unique solution to the system of equations. For the second case where $G(\cdot)$ and $R(\cdot)$ are strictly monotonic, I will show that the graphs of $G(\cdot)$ and the inverse function $R^{-1}(\cdot)$ intersect exactly once; the inverse function is defined, since $R(\cdot)$ is strictly monotonic and continuous on the domain $[0,1]$.[30]

Taking the first derivatives of $G(\cdot)$ and $R(\cdot)$ with respect to their only argument, one obtains

$$\begin{aligned}
\frac{\partial G\left(\frac{y_{43}}{y_{34}}\right)}{\partial\left(\frac{y_{21}}{y_{12}}\right)} &= a_3 a_4 \frac{[P^L+\pi_1^*(1-3)][P^L+\pi_2^*(2-4)]-[P^L+\pi_1^*(1-4)][P^L+\pi_2^*(2-3)]}{[a_3(P^L+\pi_1^*(1-3))+a_4(P^L+\pi_1^*(1-4))\left(\frac{y_{21}}{y_{12}}\right)^r]^2}\left(\frac{y_{21}}{y_{12}}\right)^{r-1} \\
\frac{\partial R\left(\frac{y_{21}}{y_{12}}\right)}{\partial\left(\frac{y_{43}}{y_{34}}\right)} &= a_1 a_2 \frac{[P^L+\pi_3^*(3-1)][P^L+\pi_4^*(4-2)]-[P^L+\pi_3^*(3-2)][P^L+\pi_4^*(4-1)]}{[a_1(P^L+\pi_3^*(3-1))+a_2(P^L+\pi_3^*(3-2))\left(\frac{y_{43}}{y_{34}}\right)^r]^2}\left(\frac{y_{43}}{y_{34}}\right)^{r-1}
\end{aligned}$$

Note that the denominator is always positive in both expressions (it is squared). This implies that the sign of the slope is fully determined by the numerator. Equilibrium requires that each of the two ratios of effort must be between zero and one. Therefore, the sign of the numerators of both $G(\cdot)$ and $R(\cdot)$ depends on a difference of two expressions of heterogeneity and prize parameters which are exogeneously given. As a consequence, one has to distinguish two cases: For both $G(\cdot)$ and $R(\cdot)$, respectively, it holds that the function is either strictly monotone in the domain of interest (increasing or decreasing), or the slope of the function is always zero. Analysis of $G(\cdot)$ reveals that the slope of $G(\cdot)$ is zero if and only if agents 1 and 2 are of the same player type; (if and only) if this is the case, it holds that $G(\cdot) = \frac{y_{21}^*}{y_{12}^*} = 1$, i.e. the two agents choose the same level of outlays. Similarly, $R(\cdot)$ is equal to one for all values of $\frac{y_{21}}{y_{12}}$ if and only if agents 3 and 4 are of the same type.

This implies that I have to consider three different scenarios for the proof: (1) $G(\cdot)$ and $R(\cdot)$ are equal to one and therefore independent of one another; (2) either $G(\cdot)$ or $R(\cdot)$ are equal to one, i.e. one of the two relations depends on the other one, but not vice versa; (3) neither $G(\cdot)$ nor $R(\cdot)$ are equal

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[29] This has been proven by Nti (1999) and Cornes & Hartley (2005), for example.

[30] Recall that the equilibrium ratios $\frac{y_{21}^*}{y_{12}^*}$ and $\frac{y_{43}^*}{y_{34}^*}$ must both be between zero and one, since it holds by assumption that (i) agent 1 is stronger than agent 2 ($a_1 \geq a_2$), and (ii) agent 3 is stronger than agent 4 ($a_3 \geq a_4$).

to one, and the two functions are interdependend. It is straightforward to show that a solution to the system consisting of (7) and (8) does exist and is unique in cases (1) and (2); case (3) is somewhat more involved and will be dealt with next.

Due to the previous reasoning, it must be the case that both $G(\cdot)$ and $R(\cdot)$ are strictly monotonic in case (3). Therefore, it is possible to determine the inverse function of $R(\cdot)$. It holds that

$$R^{-1}\left(\frac{y_{21}}{y_{12}}\right) \equiv \sqrt[r]{\frac{a_1}{a_2}\frac{[P^L+\pi_3^*(3-1)]\left(\frac{y_{43}^*}{y_{34}^*}\right)-[P^L+\pi_4^*(4-1)]}{[P^L+\pi_4^*(4-2)]-[P^L+\pi_3^*(3-2)]\left(\frac{y_{43}^*}{y_{34}^*}\right)}} = \left(\frac{y_{21}}{y_{12}}\right). \tag{21}$$

By definition of the inverse function, it must hold that $R^{-1}(\frac{y_{21}}{y_{12}})$ is strictly monotonic. Further, $R^{-1}(\frac{y_{21}}{y_{12}})$ has a unique root (in the relevant domain $\frac{y_{43}^*}{y_{34}^*} \in [0,1]$) at $Z = \frac{[P^L+\pi_4^*(4-1)]}{[P^L+\pi_3^*(3-1)]}$, where $0 < Z < 1$. Finally, close inspection of (21) reveals that $R^{-1}(\frac{y_{21}}{y_{12}})$ has a pole at $W = \frac{[P^L+\pi_4^*(4-2)]}{[P^L+\pi_3^*(3-2)]}$, $0 < W < 1$. Since $G(0)$ and $G(1)$ are both strictly smaller than 1, it must be that the graphs of the functions $G(\cdot)$ and $R^{-1}(\cdot)$ intersect exactly once in the relevant domain $\frac{y_{43}}{y_{34}} \in [0,1]$ by intermediate value theorem, which completes the proof.

Proof of Proposition 1:

This proof consists of two parts: In part (1), I will consider stage 1 winning probabilities, whereas I consider expected equilibrium payoffs in part (2). Before I can start with the proof of Proposition 1, however, I will derive the respective expressions for the stage 1 equilibrium winning probability p_{ij} and the expected equilibrium payoff in stage 1, Π_i^*. Without loss of generality, I assume that agents 1 and 2 meet in one, while agents 3 and 4 meet in the second stage 1 interaction. Then, the winning probability of agent 1 in stage 1 is defined as

$$p_{12} = \frac{a_1 y_{12}^r}{a_1 y_{12}^r + a_2 y_{21}^r} = \frac{a_1}{a_1 + a_2\left(\frac{y_{21}}{y_{12}}\right)^r}.$$

From equation (7), one can easily show that

$$\frac{y_{21}^*}{y_{12}^*} = \frac{P^L + p_{34}\times\pi_2^*(2-3) + (1-p_{34})\times\pi_2^*(2-4)}{P^L + p_{34}\times\pi_1^*(1-3) + (1-p_{34})\times\pi_1^*(1-4)} = \frac{P^L + P_2^c(y_{34},y_{43})}{P^L + P_1^c(y_{34},y_{43})}.$$

Note that $P_2^c(y_{34},y_{43})$ is increasing in a_2, while $P_1^c(y_{34},y_{43})$ is increasing in a_1. This is because the expected equilibrium payoff of a pairwise interaction between agents i and j for any agent i is strictly increasing in the effort productivity parameter of agent i, as inspection of equation (4) clearly reveals. Note that the effect of changes in a_1 or a_2 on p_{34} is ignored in this analysis; as already mentioned in the paper, the incorporation of this equilibrium reaction effect complicates the expressions to an extend that cannot be analyzed theoretically.

(1) Inserting the above expression for the effort ratio in the winning probability gives

$$p_{12} = \frac{1}{1 + \frac{a_2}{a_1}\left(\frac{P^L + P_2^c(y_{34}, y_{43})}{P^L + P_1^c(y_{34}, y_{43})}\right)^r} = \frac{1}{1+\Phi}, \tag{22}$$

where $\Phi = \frac{a_2}{a_1}(\frac{P^L + P_2^c(y_{34}, y_{43})}{P^L + P_1^c(y_{34}, y_{43})})^r$. It is straightforward to show that

$$\frac{\partial \Phi}{\partial a_1} < 0 \quad \text{and} \quad \frac{\partial \Phi}{\partial a_2} > 0 \tag{23}$$

do hold. In combination with the fact that $\frac{\partial p_{12}}{\partial \Phi} < 0$, this proves parts (a) and (b) of Proposition 1. The winning probabilities of agents 2, 3, and 4, have exactly the same structure, and proving the relations for those expressions goes through the same steps.

(2) Now, I consider the expected payoff, which can be shown to satisfy

$$\Pi_1 = \frac{1 + (1-r)\Phi}{(1+\Phi)^2}[P^L + P_1^c(y_{34}, y_{43})]. \tag{24}$$

Simple algebra shows that $\frac{\partial \Pi_1}{\partial \Phi} < 0$, which in combination with the results of (23) proves the claim. The same holds for the expected equilibrium payoffs of agents 2, 3, and 4, which have the same structure as the one for agent 1.

Proof of Proposition 2:

Without loss of generality, I assume that agents 1 and 2 meet in one, while agents 3 and 4 meet in the second stage 1 interaction. Further, agent 1 is stronger than agent 2, while agent 3 is stronger than agent 4, i.e. the relations $a_1 \geq a_2$ and $a_3 \geq a_4$ do hold. Now, I have to proof that both the overall winning probability and the expected equilibrium payoff of agents 3 and 4 are decreasing in a_1. Further, the same must hold for payoffs and probabilities of agents 1 and 2 with respect to a_3. The overall winning probability will be considered in part (1) of the proof; the expected equilibrium payoff follows in part (2).

(1) The overall winning probability for agent 3 is defined as follows:

$$\wp_3 = p_{34}^1 \times [p_{12}^1 \times p_{31}^2 + (1 - p_{12}^1) \times p_{32}^2].$$

p_{34}^1 is the probability that agent 3 wins against his stage 1 opponent 4. Conditional on winning stage 1, agent 3 meets agent 1 with probability p_{12}^1 in stage 2, and with probability p_{31}^2 he wins this stage 2 interaction. With the converse probability, agent 3 meets agent 2 in stage 2, against whom he wins with probability p^2_{32}. As already mentioned in the previous proof, I do not consider the equilibrium response with respect to stage 1 efforts that works across the two stage 1 interactions, i.e. the (extremely weak) effect of a change in a_1 on p^1_{34} is omitted. Note that $p^2_{31} \leq p^2_{32}$, i.e. agent 3 has a higher winning probability in stage 2 if he meets agent 2 (who is weaker than agent 1).

From Proposition 1 I know that p^1_{12} is increasing in a_1; p^2_{32} remains unchanged, but p^2_{31} decreases. Consequently, the overall winning probability $\wp_3$ of agent 3 is decreasing in the strength of the stronger agent in the other stage 1 interaction, a_1. Going through exactly, the same steps, one can show that the same holds for agent 4. Then, since I did not make any assumptions on the relation between agents 1 and 2 as compared to agents 3 and 4, it can be proven in the same way that the overall winning probability of agents 1 and 2 is decreasing in a_3.

(2) The expected equilibrium payoff for agent 3 is defined as

$$\Pi_3 = \frac{1 + (1-r)\frac{y_{43}}{y_{34}}}{(1+\frac{y_{43}}{y_{34}})^2}[P^L + P^c_3(y_{12}, y_{21})],$$

where $P^c_3(y_{12}, y_{21}) = p^1_{12} \times \pi^*_3(3-1) + (1-p^1_{12}) \times \pi^*_3(3-2)$. As in all previous proofs, I ignore the indirect effect across stage 1 interactions on efforts, i.e. I assume that the ratio $\frac{y_{43}}{y_{34}}$ is not affected by a change in a_1. Then, I only have to consider the effect of a change in a_1 on Π_3: From Proposition 1, I know that p^1_{12} is increasing in a_1. Further, note that $\pi^*_3(3-1) \leq \pi^*_3(3-2)$. In addition, $\pi^*_3(3-1)$ is decreasing in a_1. All those effects reduce $P^c_3(y_{12}, y_{21})$. Since Π_3 is increasing in $P^c_3(y_{12}, y_{21})$, it must hold that Π_3 is reduced if a_1 is increasing. Corresponding relations can be shown to hold for the expected payoffs agents 1, 2, and 4.

Proof of Proposition 3:

Without loss of generality, I assume that agents 1 and 2 meet in one, while agents 3 and 4 meet in the second stage 1 interaction. Further, agent 1 is stronger than agent 2, while agent 3 is stronger than agent 4, i.e. the relations $a_1 \geq a_2$ and $a_3 \geq a_4$ do hold. Now, I have to proof that both the overall winning probability and the expected equilibrium payoff of agents 3 and 4 may be increasing or decreasing in a_2. Further, the same must hold for payoffs and probabilities of agents 1 and 2 with respect to a_4. The overall winning probability will be considered in part (1) of the proof; the expected equilibrium payoff follows in part (2).

(1) Recall from the proof that the overall winning probability for agent 3 is defined as

$$\wp_3 = p^1_{34} \times [p^1_{12} \times p^2_{31} + (1-p^1_{12}) \times p^2_{32}].$$

Now, recall from Proposition 1 that p^1_{12} is decreasing in a_2; p^2_{31} remains unchanged, but p^2_{32} decreases.

Consequently, the total effect on the overall winning probability $\wp_3$ of agent 3 is ambiguos: $\wp_3$ is increasing, since p^1_{12} decreases; however, at the same time, p^2_{32} decreases, which tends to decrease $\wp_3$. Going through exactly, the same steps, one can show that the same holds for agent 4. Then, since I did not make any assumptions on the relation between agents 1 and 2 as compared to agents 3 and 4, it can be proven in the same way that the overall winning probability of agents 1 and 2 is decreasing in a_3.

(2) The expected equilibrium payoff for agent 3 is defined as

$$\Pi_3 = \frac{1 + (1-r)\frac{y_{43}}{y_{34}}}{(1+\frac{y_{43}}{y_{34}})^2}[P^L + P^c_3(y_{12}, y_{21})],$$

where $P^c_3(y_{12}, y_{21}) = p^1_{12} \times \pi^*_3(3-1) + (1-p^1_{12}) \times \pi^*_3(3-2)$. As in all previous proofs, I ignore the indirect effect across stage 1 interactions on efforts, i.e. I assume that the ratio $\frac{y_{43}}{y_{34}}$ is not affected by a change in a_2. Then, I only have to consider the effect of a change in a_2 on Π_3: From Proposition 1, I know that p^1_{12} is decreasing in a_2, which tends to increase $P^c_3(y_{12}, y_{21})$, since $\pi^*_3(3-1) \leq \pi^*_3(3-2)$. Note, however, that $\pi^*_3(3-2)$ is decreasing in a_2, an effect that tends to reduce $P^c_3(y_{12}, y_{21})$. As a consequense, the total effect of a change in a_2 on $P^c_3(y_{12}, y_{21})$ is ambigous. Since Π_3 depends linearly on $P^c_3(y_{12}, y_{21})$, the overall effect of a change in a_2 on Π_3 is unclear. Corresponding relations can be shown to hold for the expected payoffs agents 1, 2, and 4.

Proof of Proposition 4:

Proposition 4 can be proven by the presentation of two examples. Below, I present two different parametrizations for a two stage contest: Total effort provision is maximized in Seeding C: 1-2, 3-4 for the first one, while Seeding B: 1-3, 2-4 maximizes total effort provision for the second one, which proves the claim that there is no Seeding that always maximized total effort provision.

(1) Assume that the vector (a_1, a_2, a_3, a_4) is defined as follows: $(a_1, a_2, a_3, a_4) = (10, 8, 5, 1)$. Under the assumption that $r = 1$, the solution to the model that was presented in section 2.2 shows that total effort expenditures are equal to in Seeding C, to in Seeding B, and to in Seeding A, i.e. total effort provision is maximized in Seeding C: 1-2, 3-4.

(2) Assume that the vector (a_1, a_2, a_3, a_4) is defined as follows: $(a_1, a_2, a_3, a_4) = (10, 8, 5, 1)$. Under the assumption that $r = 1$, the solution to the model that was presented in section 2.2 shows that total effort expenditures are equal to in Seeding C, to in Seeding B, and to in Seeding A, i.e. total effort provision is maximized in Seeding B: 1-3, 2-4.