

10 Appendix

Case 1: All Four Areas Simultaneously

We wish to maximize the expected payoff for player X, given by

$$U_X \left(\{X_{S_r}, Y_{S_r}\}_{r=1}^4 \right) = \sum_{\alpha \in X^\star} \prod_{i \in \alpha, j \in A - \alpha} \frac{X_i Y_j}{Z_i Z_j} V - \sum_{i \in A} X_i$$

However, we can simplify this as $X_N = X_S$ and $X_E = X_W$, yielding

$$\left(\frac{X_N}{Z_N} \right) \left(\frac{X_E}{Z_E} \right)^2 + 2 \left(\frac{X_N}{Z_N} \right) \left(\frac{X_E}{Z_E} \right) \left(\frac{Y_E}{Z_E} \right) + \left(\frac{X_N}{Z_N} \right)^2 \left(\frac{Y_E}{Z_E} \right) - 2X_N - 2X_E. \quad (4)$$

Taking the derivative of this with respect to X_N yields, after some algebra, $\left(\frac{Y_N}{Z_N^2} \right) \left(\left(\frac{X_E}{Z_E} \right) \left(\frac{Y_S}{Z_S} \right) + \left(\frac{Y_W}{Z_W} \right) \left(\frac{X_S}{Z_S} \right) \right) - 1$, while taking the derivative with respect to X_E yields $\left(\frac{Y_E}{Z_E^2} \right) \left(\frac{X_N}{Z_N} \right) \left(\frac{Y_S}{Z_S} \right) - 1$. As the equations for this case are symmetric for East versus West, North versus South and because of this, X versus Y, these are the only equations required. Making this substitution and solving the system of equations for the derivatives being equal to zero gives us the system

$$\begin{aligned} \frac{1}{4X_N} \left(\frac{1}{4} + \frac{1}{4} \right) &= 1 \\ \frac{1}{4X_E} \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) &= 1 \end{aligned}$$

Solving this system of equations, we obtain $X_N = \frac{1}{8}; X_E = \frac{1}{16}$. This proves Proposition 1

Case 2 : Four areas sequentially

In order to obtain these results, we must work backwards. If only one area remains to be auctioned, taken to be East without loss of generality, then $U_X = \left(\frac{X_E}{Z_E} \right) - X_E$, and $U_Y = \left(\frac{Y_E}{Z_E} \right) - Y_E$. Taking derivatives gives us $\left(\frac{X_E}{Z_E^2} \right) = 1$, $\left(\frac{Y_E}{Z_E^2} \right) = 1$, so $X_E = Y_E = \frac{1}{4}$.

Now we can work backwards to a previous stage. Assume North and East remain to be auctioned sequentially, with Player X requiring both to win. Then $U_X = \left(\frac{X_N}{Z_N} \right) \frac{1}{4} - X_N$, $U_Y = \left(\frac{Y_N}{Z_N} \right) + \left(\frac{X_N}{Z_N} \right) \frac{1}{4} - Y_N$ gives the expected utility for each player in the North round. Taking derivatives gives us

$$\begin{aligned} \left(\frac{Y_N}{Z_N^2} \right) \frac{1}{4} &= 1 \\ \left(\frac{X_N}{Z_N^2} \right) - \left(\frac{X_N}{Z_N^2} \right) \frac{1}{4} &= \left(\frac{X_N}{Z_N^2} \right) \frac{3}{4} = 1 \end{aligned}$$

Solving this gives $X_N = \frac{3}{64}$, while $Y_N = \frac{9}{64}$, and thus expected values $U_X = \frac{1}{64}, U_Y = \frac{43}{64}$.

Subcase 2a: North or South as the first round

After the first round, one of the remaining areas will become irrelevant, with the winner of the first round needing to win one of the other two areas, and the loser of the first round needing to win both. Thus, if North is the first area, we have $U_X = \left(\frac{X_N}{Z_N}\right) \frac{43}{64} + \left(\frac{Y_N}{Z_N}\right) \frac{1}{64} - X_N$. Taking the derivative with respect to X_N gives $\left(\frac{X_N}{Z_N^2}\right) \frac{43}{64} - \left(\frac{X_N}{Z_N^2}\right) \frac{1}{64} = 1$. Due to symmetry $X_N = Y_N$, so solving gives us $X_N = Y_N = \frac{23}{128}$ and $U_X = U_Y = \frac{25}{128}$.

Subcase 2b: E-W-N-S, E-W-S-N, W-E-S-N, W-E-N-S

If player X wins the East, winning the West means he will need either the North or South, while losing the West means that the South is irrelevant, and the North will determine the overall winner. These have expected values of $\frac{43}{64}$ and $\frac{1}{4}$ respectively for player X and $\frac{1}{64}$ and $\frac{1}{4}$ for player Y. Thus, if player X wins the East, the expected value functions for the second round are

$$\begin{aligned} U_X &= \left(\frac{X_W}{Z_W}\right) \frac{43}{64} + \left(\frac{Y_W}{Z_W}\right) \frac{1}{4} - X_W \\ U_Y &= \left(\frac{Y_W}{Z_W}\right) \frac{1}{4} + \left(\frac{X_W}{Z_W}\right) \frac{1}{64} - Y_W \end{aligned}$$

This gives us first order conditions of

$$\left(\frac{Y_W}{Z_W^2}\right) \frac{27}{64} = \left(\frac{X_W}{Z_W^2}\right) \frac{15}{64} = 1$$

and so $15X_W = 27Y_W$. Thus $X_W = \frac{27}{15}Y_W$, which when placed into the first order conditions gives us $X_W = \frac{(15)(27^2)}{(42^2)(64)}$, $Y_W = \frac{(15^2)(27)}{(42^2)(64)}$. Using these values in the expected value functions gives $U_X = \frac{47907}{112896}$ and $U_Y = \frac{5139}{112896}$. These values will be reversed if player Y wins the East.

Thus, in the initial round, the expected value function for player X is $\left(\frac{X_E}{Z_E}\right) \frac{47907}{112896} + \left(\frac{Y_E}{Z_E}\right) \frac{5139}{112896} - X_E$, giving a first order condition of $\left(\frac{Y_E}{Z_E^2}\right) \frac{42768}{112896} = 1$, with $X_E = Y_E$ due to symmetry. Thus, the optimal strategy is $X_E = Y_E = \frac{42768}{451584}$, which yields expected values of $\frac{63486}{451584}$.

Subcase 2c: E-N-W-S, W-S-E-N, E-S-W-N, W-N-E-S, E-N-S-W, E-S-N-W, W-N-S-E, W-S-N-E

If player X wins the East, winning the North means victory, while losing North means he must win both West and South, for an expected value of $\frac{1}{64}$ for X and $\frac{43}{64}$ for Y. Thus the expected value functions in the second round

if X won the East are

$$U_X = \left(\frac{X_N}{Z_N}\right) + \left(\frac{Y_N}{Z_N}\right) \frac{1}{64} - X_N$$

$$U_Y = \left(\frac{Y_N}{Z_N}\right) \frac{43}{64} - Y_N$$

This gives the first order conditions of

$$\left(\frac{Y_N}{Z_N^2}\right) \frac{63}{64} = \left(\frac{X_N}{Z_N^2}\right) \frac{43}{64} = 1$$

Thus we have $43X_N = 63Y_N$, which with our first order condition gives $X_N = \frac{(43)(63^2)}{(64)(106^2)}$ and $Y_N = \frac{(43^2)(63)}{(64)(106^2)}$. Plugging these into the expected value equations give approximations of $U_X \approx .237333$, $U_Y \approx .110564$.

If player Y wins the East, winning either the South or North and West wins. This is the same as in the E-W-N-S case, which gives decimal approximations of $U_X \approx .045520$ and $U_Y \approx .424470$.

Thus for the first round, we have expected value functions of

$$U_X = \left(\frac{X_E}{Z_E}\right) .237333 + \left(\frac{Y_E}{Z_E}\right) .045520 - X_E$$

$$U_Y = \left(\frac{Y_E}{Z_E}\right) .424470 + \left(\frac{X_E}{Z_E}\right) .110564 - Y_E$$

These yield first order conditions of

$$\left(\frac{Y_E}{Z_E^2}\right) (.237333 - .045520) = \left(\frac{X_E}{Z_E^2}\right) (.424470 - .110564) = 1$$

$$\left(\frac{Y_E}{Z_E^2}\right) .191813 = \left(\frac{X_E}{Z_E^2}\right) .313906 = 1$$

$$Y_E .191813 = X_E .313906$$

$$Y_E \approx X_E 1.636521$$

Thus we find $X_E \approx .0451586$, $Y_E \approx .0739024$. Placing these in our expected value function gives $U_X \approx .0731137$, $U_Y \approx .2315069$.

As we see from these results, the total spending by the two players in each sequential case is less than .75, proving Proposition 2, and asymmetries in expected payoffs occur only in subcase 2c, in which there is an asymmetric ordering, proving Proposition 3.

Case 3: Two Rounds, Three Areas in the First Round

The case of 3 areas being auctioned first, followed by the single remaining area, is also relatively simple to solve. We will work backwards, as in the sequential case.

This case can be subdivided into two subcases, one where North and South are both in the initial group of 3 areas, and one where East and West are both in the initial group.

Subcase 3a: North and South in the initial group

We will consider the case where North, South, and West are the original group. Results for North, South, and East as the initial group differ only in switching the amounts invested in the East and West areas with each other.

Now when we consider the competition for the original 3 areas, we have the required information. Completing a winning path at this stage will result in a value of 1, losing at this stage will have a value of 0, and a draw, which will be resolved with the final area has an expected value of $\frac{1}{4}$. Thus we can compute optimal strategy for the original 3 areas of competition. We do so by once again calculating the expected value of the contest and taking the derivatives. The expected value for player X, after a bit of algebra, is

$$\left(\left(\frac{X_W}{Z_W}\right)\left(\frac{X_S}{Z_S}\right) + \left(\frac{Y_W}{Z_W}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{X_N}{Z_N}\right)\right) + \left(\left(\frac{Y_S}{Z_S}\right)\left(\frac{X_N}{Z_N}\right)\right)\left(\frac{1}{4}\right) - X_W - X_S - X_N$$

Taking the derivatives and setting them equal to zero gives the following system of equations:

$$\begin{aligned} \left(\left(\frac{1}{Z_W} - \left(\frac{X_W}{Z_W^2}\right)\right) - \left(\frac{Y_W}{Z_W^2}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{X_N}{Z_N}\right)\right) &= 1 \\ \left(\frac{Y_W}{Z_W}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{1}{Z_N} - \left(\frac{X_N}{Z_N}\right)\right) + \frac{1}{4}\left(\frac{Y_S}{Z_S}\right)\left(\frac{1}{Z_N} - \left(\frac{X_N}{Z_N^2}\right)\right) &= 1 \\ \left(\frac{X_W}{Z_W}\right)\left(\frac{Z_S - X_S}{Z_S^2}\right) + \left(\frac{Y_W}{Z_W}\right)\left(\frac{Z_S - X_S}{Z_S^2}\right)\left(\frac{X_N}{Z_N}\right) - \frac{1}{4}\left(\frac{Y_S}{Z_S^2}\right)\left(\frac{X_N}{Z_N}\right) &= 1 \end{aligned} \tag{5}$$

As seen in the diagram of our game, reflection across the horizontal or vertical axis will interchange the players. For example, player X faces the same decision for the North area as player Y does for the South. Thus, we know $X_N = Y_S, X_S = Y_N, X_W = Y_W$ in equilibrium. This allows us, after some algebra, to solve this system of equations, with the optimal values

$$\begin{aligned} X_S = Y_N &= \frac{18}{125} \\ X_N = Y_S &= \frac{12}{125} \\ X_W = Y_W &= \frac{9}{100} \end{aligned}$$

Thus, both players have a probability of victory of $\frac{1}{2}$, as they have symmetric strategies, and each will thus have an expected value from this game of $\frac{13}{100}$.

Subcase 3b: East and West in the initial group

Again, we will use one of the two cases to illustrate the solution, as the solutions will differ only by symmetry. The following solution uses East, West, and North as the original group.

If player X wins areas North and East in the initial round, he will win the prize at that point. Similarly, if player Y wins West and North in the initial round, this results in immediate victory. If neither of these occur, South will be contested for the entire prize. If this second round happens, we know both players have an expected value of $\frac{1}{4}$. Thus the expected value for player X can be given by

$$\begin{aligned} & \left(\frac{X_E}{Z_E}\right) \left(\frac{X_N}{Z_N}\right) + \left(1 - \left(\frac{X_E}{Z_E}\right) \left(\frac{X_N}{Z_N}\right) - \left(\frac{Y_W}{Z_W}\right) \left(\frac{Y_N}{Z_N}\right)\right) - X_N - X_W - X_E \\ &= \frac{1}{4} + \left(\frac{X_E}{Z_E}\right) \left(\frac{X_N}{Z_N}\right) \frac{3}{4} - \left(\frac{Y_W}{Z_W}\right) \left(\frac{Y_N}{Z_N}\right) \frac{1}{4} - X_N - X_W - X_E \end{aligned}$$

Taking the derivatives and setting them equal to zero will thus give us the following first order conditions

$$\begin{aligned} & \left(\frac{Y_E}{Z_E^2}\right) \left(\frac{X_N}{Z_N}\right) \frac{3}{4} = 1 \\ & \left(\frac{Y_W}{Z_W^2}\right) \left(\frac{Y_N}{Z_N}\right) \frac{1}{4} = 1 \\ & \left(\frac{X_E}{Z_E}\right) \left(\frac{Y_N}{Z_N^2}\right) \frac{3}{4} + \left(\frac{Y_W}{Z_W}\right) \left(\frac{Y_N}{Z_N^2}\right) \frac{1}{4} = 1 \end{aligned}$$

Similarly, player Y will have the expected value given by

$$\begin{aligned} & \left(\frac{Y_W}{Z_W}\right) \left(\frac{Y_N}{Z_N}\right) + \left(1 - \left(\frac{X_E}{Z_E}\right) \left(\frac{X_N}{Z_N}\right) - \left(\frac{Y_W}{Z_W}\right) \left(\frac{Y_N}{Z_N}\right)\right) - Y_N - Y_W - Y_E \\ &= \frac{1}{4} + \left(\frac{Y_W}{Z_W}\right) \left(\frac{Y_N}{Z_N}\right) \frac{3}{4} - \left(\frac{X_E}{Z_E}\right) \left(\frac{X_N}{Z_N}\right) \frac{1}{4} - Y_N - Y_W - Y_E \end{aligned}$$

This yields the first order conditions of

$$\begin{aligned} & \left(\frac{X_E}{Z_E^2}\right) \left(\frac{X_N}{Z_N}\right) \frac{1}{4} = 1 \\ & \left(\frac{X_W}{Z_W^2}\right) \left(\frac{Y_N}{Z_N}\right) \frac{3}{4} = 1 \\ & \left(\frac{Y_W}{Z_W}\right) \left(\frac{X_N}{Z_N^2}\right) \frac{3}{4} + \left(\frac{X_E}{Z_E}\right) \left(\frac{X_N}{Z_N^2}\right) \frac{1}{4} = 1 \end{aligned}$$

Using these sets of first order conditions along with making the assumption that $X_N = Y_N$, justified by North occupying the same importance for both players, allows us to solve for optimal first round strategies of

$$\begin{aligned} X_N &= Y_N = \frac{3}{16} \\ X_W &= Y_E = \frac{3}{128} \\ X_E &= Y_W = \frac{9}{128} \end{aligned}$$

Using these values in our expected value equations gives an expected value for each player of $\frac{5}{32}$. Each player will on average spend $\frac{11}{16}$, and both have a probability of victory of $\frac{1}{2}$.

Case 4: Two Rounds, Two Areas in Each

This case will require a solution to the expected values of each player if only 2 areas remain, they are being auctioned simultaneously, and one player requires both areas for victory, while the other player requires either for victory. For notation, we will use player X as needing both the North and East areas. If player X needs both areas, his expected value is $\left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right) - X_N - X_E$, while player Y has an expected value of $1 - \left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right) - Y_N - Y_E$. Taking derivatives, we obtain the following set of first order conditions

$$\left(\frac{Y_N}{Z_N^2}\right)\left(\frac{X_E}{Z_E}\right) = \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_E}{Z_E^2}\right) = \left(\frac{X_N}{Z_N^2}\right)\left(\frac{X_E}{Z_E}\right) = \left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E^2}\right) = 1 \quad (6)$$

After a bit of algebra, we obtain that $X_N = Y_N = X_E = Y_E = \frac{1}{8}$, so player X has an expected value of 0, while player Y has an expected value of $\frac{1}{2}$.

We now have enough information to write the expected value equations and thus solve this case. However, we will need to break this case into three subcases.

Subcase 4a: East and West as the First Round If player X wins both the East and West in the initial round, they will complete a winning set with either North or South, and thus X will have an expected value of $\frac{1}{2}$, while Y will have an expected value of 0. If players X and Y split East and West, only one of North and South will matter, and thus each player has an expected value of $\frac{1}{4}$. Thus, player X has an expected value of

$$\left(\frac{X_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{2} + \left(\frac{X_E}{Z_E}\right)\left(\frac{Y_W}{Z_W}\right)\frac{1}{4} + \left(\frac{Y_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{4} - X_E - X_W$$

As East and West are symmetric, as well as players X and Y, we know that $X_E = X_W = Y_E = Y_W$. Thus, taking the derivative with respect to X_E gives us the following

$$\begin{aligned} \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{2} + \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{Y_W}{Z_W}\right)\frac{1}{4} - \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{4} - 1 = \\ \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{Y_E}{Z_E}\right)\frac{1}{2} - 1 = \\ \left(\frac{Y_E}{Z_E^2}\right)\frac{1}{4} - 1 = \\ \frac{1}{4X_E}\frac{1}{4} - 1 \end{aligned}$$

Setting this equal to zero gives us first order conditions of $X_E = X_W = Y_E = Y_W = \frac{1}{16}$, and using this gives us expected values for each player of $\frac{1}{8}$.

Subcase 4b : North and South as the First Round

This subcase is very similar to the previous subcase. The difference is that if one player wins both North and South in the initial round, this constitutes a winning set without need for a second round. Thus, player X has an expected value function of

$$\left(\frac{X_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right) + \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\frac{1}{4} + \left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\frac{1}{4} - X_N - X_S$$

Again, due to the symmetries of both players and North and South, we have variables know to be the same, with $X_E = X_W = Y_E = Y_W$. Thus, we take the derivative with respect to X_N , and obtain

$$\left(\frac{Y_N}{Z_N^2}\right)\left(\frac{X_S}{Z_S}\right) + \left(\frac{Y_N}{Z_N^2}\right)\left(\frac{Y_S}{Z_S}\right)\frac{1}{4} - \left(\frac{Y_N}{Z_N^2}\right)\left(\frac{X_S}{Z_S}\right)\frac{1}{4} - 1 = \frac{1}{4X_N} - 1$$

Setting this first order condition equal to zero, we see that $X_N = X_S = Y_N = Y_S = \frac{1}{8}$. Placing this into our expected value function, we find that both players have an expected value of $\frac{1}{8}$. Thus subcases 3a and 3b have the same expected value despite having different optimal strategies.

Subcase 4c: All other 2 areas followed by 2 areas cases This case illustrates the asymmetries that can occur during this game. There will be 2 rounds of auctions, each contesting 2 areas. Up to symmetry, these cases all work the same, so without loss of generality we will assume that in the first round, the West and South areas are auctioned, in the second round the North and East are available. When the second round begins, 3 different situations are possible: X may have won in the first round, X and Y may both require North, with East irrelevant, or X may require both North and East, with Y requiring only one of these two areas for victory.

If X has already won, the auctions in the second round effectively may be ignored. If both players require only North, player X will have an expected return of $\left(\frac{X_N}{Z_N}\right) - X_N$, which, when the derivative with respect to X_N is taken results in $\left(\frac{Y_N}{Z_N^2}\right) = 1$. We know from symmetry $X_N = Y_N$, giving us an equilibrium of $X_N = Y_N = \frac{1}{4}$, and an expected value for each player of $\frac{1}{4}$ at this stage, as seen in case 2. Now we have enough information to find the equilibrium for the first round. The expected values are

$$E[V_X] = \left(\frac{X_W}{Z_W}\right)\left(\frac{X_S}{Z_S}\right) + \left(\frac{Y_W}{Z_W}\right)\left(\frac{X_S}{Z_S}\right)\frac{1}{4} - X_S - X_W$$

$$E[V_Y] = \left(\frac{Y_S}{Z_S}\right)\frac{1}{2} + \left(\frac{Y_W}{Z_W}\right)\left(\frac{X_S}{Z_S}\right)\frac{1}{4} - Y_S - Y_W$$

Taking the derivatives gives us a set of equilibrium equations

$$\begin{aligned} \left(\frac{Y_W}{Z_W}\right) \left(\frac{X_S}{Z_S}\right) - \left(\frac{Y_W}{Z_W^2}\right) \left(\frac{X_S}{Z_S}\right) \frac{1}{4} &= \\ \left(\frac{X_W}{Z_W}\right) \left(\frac{Y_S}{Z_S^2}\right) + \left(\frac{Y_W}{Z_W}\right) \left(\frac{Y_S}{Z_S^2}\right) \frac{1}{4} &= \\ \left(\frac{X_W}{Z_W^2}\right) \left(\frac{X_S}{Z_S}\right) \frac{1}{4} = \left(\frac{X_S}{Z_S^2}\right) \frac{1}{2} - \left(\frac{Y_W}{Z_W}\right) \left(\frac{X_S}{Z_S^2}\right) \frac{1}{4} &= 1 \end{aligned}$$

From this point, we can solve these equations to obtain optimal behavior in the first round of $X_S = \frac{125}{2304}$, $Y_S = \frac{625}{16128}$, $X_W = \frac{21}{262}$, $Y_W = \frac{7}{262}$. Plugging these values into our expected value equations gives us approximate values of the game for player X being .3396, and for player Y, .1796. Thus, we have an asymmetric game. This is due to the differing complementarities, with the order information is released favoring player X.

Case 5: Two Rounds, One Area in the First Round

Subcase 5a: E-NWS, W-NES

If player X wins the East, winning either North or West and South gives victory, while player Y must win the North and either of the West and South. Thus the expected values for the players are

$$\begin{aligned} EV[X] &= \left(\frac{X_N}{Z_N}\right) + \left(\frac{Y_N}{Z_N}\right) \left(\frac{X_W}{Z_W}\right) \left(\frac{X_S}{Z_S}\right) - X_N - X_W - X_S \\ EV[Y] &= \left(\frac{Y_N}{Z_N}\right) \left(1 - \left(\frac{X_W}{Z_W}\right) \left(\frac{X_S}{Z_S}\right)\right) - Y_N - Y_W - Y_S \end{aligned}$$

Taking derivatives and setting them equal to zero gives first order conditions of

$$\begin{aligned} \left(\frac{Y_N}{Z_N^2}\right) - \left(\frac{Y_N}{Z_N^2}\right) \left(\frac{X_S}{Z_S}\right) \left(\frac{X_W}{Z_W}\right) &= 1 \\ \left(\frac{Y_N}{Z_N}\right) \left(\frac{Y_W}{Z_W^2}\right) \left(\frac{X_S}{Z_S}\right) &= 1 \\ \left(\frac{Y_N}{Z_N}\right) \left(\frac{X_W}{Z_W}\right) \left(\frac{Y_S}{Z_S^2}\right) &= 1 \\ \left(\frac{X_N}{Z_N^2}\right) \left(1 - \left(\frac{X_W}{Z_W}\right) \left(\frac{X_S}{Z_S}\right)\right) &= 1 \\ \left(\frac{Y_N}{Z_N}\right) \left(\frac{X_W}{Z_W^2}\right) \left(\frac{X_S}{Z_S}\right) &= 1 \\ \left(\frac{Y_N}{Z_N}\right) \left(\frac{X_W}{Z_W}\right) \left(\frac{X_S}{Z_S^2}\right) &= 1 \end{aligned}$$

From the last two equations we see that $X_S + Y_S = X_W + Y_W$. This, combined with the second, third, and fifth equations gives us that $X_W = X_S = Y_W = Y_S$, which then gives us $X_N = Y_S$. $X_S = \frac{1}{16}$ and $X_N = \frac{1}{12}$, giving an expected value for player X of $\frac{5}{12}$ and player Y and expected value of $\frac{1}{6}$.

Subcase 5b: N-EWS, S-EWN

Depending on the winning of the first round, either East or West will become irrelevant. The winner of the North needs to win either of the remaining relevant areas, which has an expected value of $\frac{1}{2}$, while the loser must win both, with expected value 0. Thus the expected value for player X is $\left(\frac{X_N}{Z_N}\right)\frac{1}{2} - X_N$, which gives a first order condition of $\left(\frac{Y_N}{Z_N^2}\right)\frac{1}{2} = 1$. As players X and Y face the same situation, they will play identical strategies, so $X_N = Y_N$. Thus solving the first order condition gives $X_N = Y_N = \frac{1}{8}$, so the expected value for each player is $\frac{1}{8}$.

Case 6: Three Rounds, Two Areas in the First Round

There are, up to symmetry, 3 distinct subcases of the three round structure where two areas are contested in the first round.

Subcase 6a: NS-E-W, NS-W-E

As no matter the result on the competition for the North and South, only one of East and West matter, this is identical to the main case of NS-EW, and thus has the same expected values of $\frac{1}{8}$ for each player.

Subcase 6b: EW-N-S, EW-S-N

This is very similar to the already solved EW-NS case, however the sequential end game changes the expected values, giving the values for player X as

$$\left(\frac{X_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right)\frac{43}{64} + \left(\frac{X_E}{Z_E}\right)\left(\frac{Y_W}{Z_W}\right)\frac{1}{4} + \left(\frac{Y_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{4} + \left(\frac{Y_E}{Z_E}\right)\left(\frac{Y_W}{Z_W}\right)\frac{1}{64} - X_E - X_W$$

Taking the derivative with respect to X_E yields

$$\left(\frac{Y_E}{Z_E^2}\right)\left(\frac{X_W}{Z_W}\right)\frac{27}{64} + \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{Y_W}{Z_W}\right)\frac{15}{64} - 1$$

As we know by symmetry that $X_E = X_W = Y_E = Y_W$, this derivative simplifies to

$$\frac{1}{8X_E}\frac{27}{64} + \frac{1}{8X_E}\frac{15}{64} = 1$$

Setting this first order condition equal to zero gives $X_E = \frac{21}{256} = X_W = Y_E = Y_W$, which placed into the expected value equation gives the expected value for each player as $\frac{17}{128}$.

Subcase 6c: NE-S-W, NE-W-S, SW-N-E, SW-E-N, NW-S-E, NW-E-S, SE-N-W, SE-W-N

These 8 cases behave the same, up to symmetry. Unfortunately, unlike the other 2-1-1 cases, we cannot make simplifying assumptions of symmetric behavior between players or between areas. Thus we need the expected

value functions for both players X and Y. We will use the NE-S-W case to demonstrate.

$$EV[X] = \left(\frac{X_N}{Z_N}\right) \left(\frac{X_E}{Z_E}\right) + \left(\frac{X_N}{Z_N}\right) \left(\frac{Y_E}{Z_E}\right) \frac{1}{4} + \left(\frac{Y_N}{Z_N}\right) \frac{1}{64} - X_N - X_E$$

$$EV[Y] = \left(\frac{X_N}{Z_N}\right) \left(\frac{Y_E}{Z_E}\right) \frac{1}{4} + \left(\frac{Y_N}{Z_N}\right) \frac{43}{64}$$

Taking the derivatives of these expected value functions against the investment in the North and East gives us the following first order conditions

$$\begin{aligned} & \left(\frac{Y_N}{Z_N^2}\right) \left(\frac{X_E}{Z_E}\right) + \left(\frac{Y_N}{Z_N^2}\right) \left(\frac{Y_E}{Z_E}\right) \frac{1}{4} - \left(\frac{Y_N}{Z_N^2}\right) \frac{1}{64} - 1 \\ & \left(\frac{X_N}{Z_N}\right) \left(\frac{Y_E}{Z_E^2}\right) - \left(\frac{X_N}{Z_N}\right) \left(\frac{Y_E}{Z_E^2}\right) \frac{1}{4} - 1 \\ & - \left(\frac{X_N}{Z_N^2}\right) \left(\frac{Y_E}{Z_E}\right) \frac{1}{4} + \left(\frac{X_N}{Z_N^2}\right) \frac{43}{64} - 1 \\ & \left(\frac{X_N}{Z_N}\right) \left(\frac{X_E}{Z_E^2}\right) \frac{1}{4} - 1 \end{aligned}$$

Setting these equal to zero, we begin solving. The second and fourth combine to give us that $3Y_E = X_E$. Thus $X_E = \frac{1}{4}$, while $Y_E = \frac{3}{4}$. Substituting this into the first and third conditions gives us $\left(\frac{Y_N}{Z_N^2}\right) \frac{51}{64} = \left(\frac{X_N}{Z_N^2}\right) \frac{47}{64}$. Thus $51Y_N = 47X_N$, which gives us values of $X_N = \frac{(51^2)(47)}{(98^2)(64)}$ and $Y_N = \frac{(51)(47^2)}{(98^2)(64)}$. Using these values in the fourth first order condition gives us $X_E = \frac{(51)(9)}{(98)(64)}$, which leads to $Y_E = \frac{(51)(3)}{(98)(64)}$. Substituting into the expected value equations gives us an answer of

$$EV[X] = \frac{51}{98} \frac{3}{4} + \frac{51}{98} \frac{1}{4} \frac{1}{4} + \frac{47}{98} \frac{1}{64} - \frac{(51^2)(47)}{(98)(64)} - \frac{(51)(9)}{(98)(64)}$$

$$EV[Y] = \frac{51}{98} \frac{1}{4} \frac{1}{4} + \frac{47}{98} \frac{43}{64} - \frac{(51)(47^2)}{(98)(64)} - \frac{(51)(3)}{(98)(64)}$$

Case 7: Three Rounds, Two Areas in the Second Round

Subcase 7a: N-EW-S, S-EW-N

After the first area is contested, either East or West will be irrelevant. Winning the first round means that winning one of the two remaining rounds is required, with an expected value of $\frac{43}{64}$, while losing means both are required, for an expected value of $\frac{1}{64}$, so player X has an expected value of $\left(\frac{X_N}{Z_N}\right) \frac{43}{64} + \left(\frac{Y_N}{Z_N}\right) \frac{1}{64} - X_N = \left(\frac{X_N}{Z_N}\right) \frac{42}{64} + \frac{1}{64} - X_N$. This gives a first order condition of $\left(\frac{Y_N}{Z_N^2}\right) \frac{42}{64} = 1$, and as we know that $X_N = Y_N$, this gives us $\frac{1}{4X_N} \frac{42}{64} = 1$, so $X_N = Y_N = \frac{23}{128}$, giving us an expected value of $\frac{25}{128}$ for each player.

Subcase 7b: N-ES-W, S-WN-E, N-WS-E, S-EN-W

As usual, we will take the first as our example without loss of generality. If player X wins the North, West is irrelevant, making X's expected value $\frac{1}{2}$ and Y's expected value 0. If Y wins the North, East is irrelevant, making X's expected value $\frac{1}{64}$ and Y's expected value $\frac{43}{64}$. Thus we have the expected values of

$$\begin{aligned} EV[X] &= \left(\frac{X_N}{Z_N}\right) \frac{1}{2} + \left(\frac{Y_N}{Z_N}\right) \frac{1}{64} - X_N \\ EV[Y] &= \left(\frac{Y_N}{Z_N}\right) \frac{43}{64} - Y_N \end{aligned}$$

Taking the first order conditions, we have

$$\begin{aligned} \left(\frac{Y_N}{Z_N^2}\right) \frac{31}{64} &= 1 \\ \left(\frac{X_N}{Z_N^2}\right) \frac{43}{64} &= 1 \end{aligned}$$

Thus we have $31Y_N = 43X_N$, and plugging this into the first order conditions gives us $Y_N = \frac{(43^2)(31)}{(74^2)(64)}$, $X_N = \frac{(43)(31^2)}{(74^2)(64)}$. We finally place this into the expected value equations, yielding $EV[X] = \frac{31}{74} \frac{1}{2} + \frac{43}{74} \frac{1}{64} - \frac{(43)(31^2)}{(74^2)(64)}$, $EV[Y] = \frac{43}{74} \frac{43}{64} - \frac{(43^2)(31)}{(74^2)(64)}$.

Subcase 7c: E-NS-W, W-NS-E

If player X wins the East, he will win with either winning the North or both South and West, while Y will need both North and either South or West. Thus, the expected values if X wins the East are

$$\begin{aligned} EV[X] &= \left(\frac{X_N}{Z_N}\right) + \left(\frac{Y_N}{Z_N}\right) \left(\frac{X_S}{Z_S}\right) \frac{1}{4} - X_N - X_S \\ EV[Y] &= \left(\frac{Y_N}{Z_N}\right) \left(\frac{Y_S}{Z_S}\right) + \left(\frac{Y_N}{Z_N}\right) \left(\frac{X_S}{Z_S}\right) \frac{1}{4} - Y_N - Y_S \end{aligned}$$

Taking derivatives and setting them equal to zero gives us the first order conditions

$$\begin{aligned} \left(\frac{Y_N}{Z_N^2}\right) - \left(\frac{Y_N}{Z_N}\right) \left(\frac{X_S}{Z_S}\right) \frac{1}{4} &= 1 \\ \left(\frac{Y_N}{Z_N}\right) \left(\frac{Y_S}{Z_S^2}\right) \frac{1}{4} &= 1 \\ \left(\frac{X_N}{Z_N^2}\right) \left(\frac{Y_S}{Z_S}\right) + \left(\frac{X_N}{Z_N^2}\right) \left(\frac{X_S}{Z_S}\right) \frac{1}{4} &= 1 \\ \left(\frac{Y_N}{Z_N}\right) \left(\frac{X_S}{Z_S^2}\right) - \left(\frac{Y_N}{Z_N}\right) \left(\frac{X_S}{Z_S}\right) \frac{1}{4} &= 1 \end{aligned}$$

The second and fourth conditions combine to give us $3X_S = Y_S$. Plugging this into the first and third conditions gives us $7X_N = 15Y_N$. Thus we have

the following optimal strategies for both players

$$\begin{aligned} X_N &= \frac{(7)(15^2)}{(22^2)(16)} \\ Y_N &= \frac{(15^3)}{(22^2)(16)} \\ X_S &= \frac{21}{(22)(64)} \\ Y_S &= \frac{63}{(22)(64)} \end{aligned}$$

Placing these values into the expected value equations give us approximate expected values for player X of .453577 and for player Y of .164482. These values are reversed if player Y wins the East, so the expected value of winning the East is the difference between these values. Thus the original expected value for player X is given by $\left(\frac{X_E}{Z_E}\right) .453577 + \left(\frac{Y_E}{Z_E}\right) .164482 - X_E$, giving a first order condition of $\left(\frac{Y_E}{Z_E}\right) .453577 - \left(\frac{Y_E}{Z_E}\right) .164482 = 1$. Player Y will have an identical strategy, thus making the optimal strategy to invest $\frac{1}{4}(.453577 - .164482)$ in the East, giving the expected value for each player of approximately .2368.

Subcase 7d: E-NW-S, W-SE-N, E-SW-N, W-NE-S

Taking the first case without loss of generality, if X wins the East, we see the same expected values as in the E-NS-W case when X won the East. However, if Y wins the East, X will have to win either of the North or West, and then must win the South. Thus, the expected values are

$$\begin{aligned} EV[X] &= \left(1 - \left(\frac{Y_N}{Z_N}\right) \left(\frac{Y_W}{Z_W}\right)\right) \frac{1}{4} - X_N - X_W \\ EV[Y] &= \left(\frac{Y_N}{Z_N}\right) \left(\frac{Y_W}{Z_W}\right) + \left(\frac{Y_N}{Z_N}\right) \left(\frac{X_W}{Z_W}\right) \frac{1}{4} + \left(\frac{X_N}{Z_N}\right) \left(\frac{Y_W}{Z_W}\right) \frac{1}{4} - Y_N - Y_W \end{aligned}$$

These give us the first order conditions

$$\begin{aligned} \left(\frac{Y_N}{Z_N^2}\right) \left(\frac{Y_W}{Z_W}\right) \frac{1}{4} &= 1 \\ \left(\frac{Y_N}{Z_N}\right) \left(\frac{Y_W}{Z_W^2}\right) \frac{1}{4} &= 1 \\ \left(\frac{X_N}{Z_N^2}\right) \left(\frac{Y_W}{Z_W}\right) + \left(\frac{X_N}{Z_N^2}\right) \left(\frac{X_W}{Z_W}\right) \frac{1}{4} - \left(\frac{X_N}{Z_N^2}\right) \left(\frac{Y_W}{Z_W}\right) \frac{1}{4} &= 1 \\ \left(\frac{Y_N}{Z_N}\right) \left(\frac{X_W}{Z_W^2}\right) - \left(\frac{Y_N}{Z_N}\right) \left(\frac{X_W}{Z_W}\right) \frac{1}{4} + \left(\frac{X_N}{Z_N}\right) \left(\frac{X_W}{Z_W^2}\right) \frac{1}{4} &= 1 \end{aligned}$$

The first two conditions give us $X_N + Y_N = X_W + Y_W$. This, combined with the third and fourth conditions gives us the following

$$\begin{aligned} X_N Y_W + \frac{1}{4} X_N X_W - \frac{1}{4} X_N Y_W &= Y_N X_W - \frac{1}{4} Y_N X_W + \frac{1}{4} X_N X_W \\ X_N Y_W &= Y_N X_W \\ Y_W (X_W + Y_W - Y_N) &= Y_N X_W \\ X_N = Y_N = X_W = Y_W &= \frac{1}{32} \end{aligned}$$

This gives us expected values for player X of $\frac{1}{8}$ and for player Y of $\frac{5}{16}$. Thus the value for player X of winning the East is $.453577 - .125 \approx .328577$, while the value for player Y of winning the East is $.3125 - .164482 \approx .148018$. This gives first order conditions of $\left(\frac{X_E}{Z_E^2}\right) .148081 \approx \left(\frac{Y_E}{Z_E^2}\right) .328577 \approx 1$. Thus $X_E .148081 \approx Y_E .328577$, giving us $X_E \approx 3.2189005 Y_E$. Thus $\frac{1}{X_E 10.36132} \approx \frac{1}{.148081}$, so we have $X_E \approx .014292$, $Y_E \approx .006441$, giving us our expected values of

$$\begin{aligned} EV[X] &\approx \frac{.014292}{.014292 + .006441} .328577 + .125 - .014292 \approx .337208 \\ EV[Y] &\approx \frac{.006441}{.014292 + .006441} .148018 + .164482 - .006441 \approx .204025 \end{aligned}$$

Case 8: Three Rounds, Two Ares in the Third Round

Subcase 8a: N-S-EW, S-N-EW

This works identically to N-EW-S, as winning North leaves needing to win one of two consecutive competitions, while losing leaves needing to win both. Thus, the same expected value for each player of $\frac{23}{128}$.

Subcase 8b: N-E-SW, S-W-NE, N-W-SE, S-E-NW

This works the same as N-SW-E, as after North is resolved, either East or West has become irrelevant. Thus we have the expected values $EV[X] = \frac{43}{74} \frac{43}{64} - \frac{(43^2)(31)}{(74^2)(64)}$, $EV[Y] = \frac{31}{74} \frac{1}{2} + \frac{43}{74} \frac{1}{64} - \frac{(43)(31^2)}{(74^2)(64)}$.

Subcase 8c: E-N-SW, W-S-NE, E-S-NW, W-N-SE

If player X wins the East, winning the North gives victory, while losing the North leaves needing to win both South and West. Thus if X wins the East, we have expected values of

$$\begin{aligned} EV[X] &= \left(\frac{X_N}{Z_N}\right) - X_N \\ EV[Y] &= \left(\frac{Y_N}{Z_N}\right) \frac{1}{2} - Y_N \end{aligned}$$

This gives us first order conditions of $\left(\frac{Y_N}{Z_N^2}\right) = 1$, $\left(\frac{X_N}{Z_N^2}\right) \frac{1}{2} = 1$, so $X_N = \frac{2}{9}$, $Y_N = \frac{1}{9}$, giving expected values of $EV[X] = \frac{4}{9}$, $EV[Y] = \frac{1}{18}$.

If player Y wins the East, player X winning the North will make only the South relevant, giving both players an expected value of $\frac{1}{4}$, while player Y winning the North as well will force player X to win both the South and West, making $EV[X] = 0$, $EV[Y] = \frac{1}{2}$. Thus the value of the North will be $\frac{1}{4}$ for each player, so they will each invest $\frac{1}{16}$, making $EV[X] = \frac{1}{16}$, $EV[Y] = \frac{5}{16}$.

Now we calculate the expected values for each player during the initial

round.

$$EV[X] = \left(\frac{X_E}{Z_E}\right) \frac{4}{9} + \left(\frac{Y_E}{Z_E}\right) \frac{1}{16} - X_E$$

$$EV[Y] = \left(\frac{Y_E}{Z_E}\right) \frac{5}{16} + \left(\frac{X_E}{Z_E}\right) \frac{1}{18} - Y_E$$

This gives us the first order conditions $\left(\frac{Y_E}{Z_E}\right) \frac{119}{288} = \left(\frac{X_E}{Z_E}\right) \frac{74}{288} = 1$, so we know that $74X_E = 119Y_E$. Plugging this into the first order conditions gives $X_E = \frac{(74)(119^2)}{(288)(193^2)}$, $Y_E = \frac{(74^2)(119)}{(288)(193^2)}$, which when placed into the expected value equations gives $EV[X] \approx .200317$, $EV[Y] \approx .093329$.

Subcase 8d: E-W-NS, W-E-NS

If players X and Y split the East and West, only one of North and South matter, and so each will have an expected value of $\frac{1}{4}$. If one player wins both East and West, the other must win both North and South simultaneously, making the expected values $\frac{1}{2}$ and 0. Thus winning each of East and West increase expected value by $\frac{1}{4}$, so each player will invest $\frac{1}{16}$ in each. As all competitions have identical strategies, each player will have a probability of victory of $\frac{1}{2}$, thus spending $\frac{1}{8}$ total in the first two rounds and averaging an expected value of $\frac{1}{4}$ in the final round for a total expected value of $\frac{1}{8}$.

Type	Order	$E[U_X]$	$E[U_Y]$	EV[A]
4	NESW	.125	.125	.75
3-1	NSW-E, NSE-W	.13	.13	.74
3-1	EWN-S, EWS-N	.1563	.1563	.6875
2-2	NE-WS, WS-NE	.3396	.1793	.4811
2-2	SE-NW, NW-SE	.1793	.3396	.4811
2-2	NS-EW	.125	.125	.75
2-2	EW-NS	.125	.125	.75
1-3	N-ESW, S-ENW	.125	.125	.75
1-3	E-NSW, W-NSE	.2292	.2292	.5417
2-1-1	EW-N-S, EW-S-N	.1328	.1328	.7344
2-1-1	NE-S-W, SW-N-E, NE-W-S, SW-E-N	.1583	.1471	.6947
2-1-1	NW-S-E, SE-N-W, NW-E-S, SE-W-N	.1471	.1583	.6947
2-1-1	NS-E-W, NS-W-E	.125	.125	.75
1-2-1	N-EW-S, S-EW-N	.1797	.1797	.6719
1-2-1	N-ES-W, S-WN-E	.1006	.2269	.6725
1-2-1	S-EN-W, N-WS-E	.2269	.1006	.6725
1-2-1	E-NS-W, W-NS-E	.2368	.2368	.5265
1-2-1	E-NW-S, W-SE-N	.2040	.3372	.4588
1-2-1	W-NE-S, E-SW-N	.3372	.2040	.4588
1-1-2	N-S-EW, S-N-EW	.1797	.1797	.6719
1-1-2	N-E-SW, S-W-NE	.1006	.2269	.6725
1-1-2	N-W-SE, S-E-NW	.2269	.1006	.6725
1-1-2	E-S-NW, W-N-SE	.0933	.2003	.7064
1-1-2	E-N-SW, W-S-NE	.2003	.0933	.7064
1-1-2	E-W-NS, W-E-NS	.125	.125	.75
1-1-1-1	N-(E,W,S), S-(E,W,N)	.1797	.1797	.6719
1-1-1-1	E-W-N-S, E-W-S-N, W-E-N-S, W-E-S-N	.1406	.1406	.7188
1-1-1-1	E-N-W-S, W-S-E-N, E-N-S-W, W-S-N-E	.0731	.2315	.6954
1-1-1-1	W-N-E-S, E-S-W-N, W-N-S-E, E-S-N-W	.2315	.0731	.6954

Table 7: Calculated Expected Values For Each Player and Auctioneer Under All Structures